# Combinatorics II (Lecture 37)

March 3, 2015

We retain the notational conventions of the previous lecture. Our goal is to prove the following result, which we asserted last time without proof:

**Lemma 1.** *The maps* $\mathrm{N}_\bullet(\mathcal{E}^\circ_\bullet) \to \mathrm{N}_\bullet(\mathcal{D}^\circ_\bullet)$ *and* $\mathrm{N}_\bullet(\mathcal{E}_\bullet) \to \mathrm{N}_\bullet(\mathcal{D}_\bullet)$ *are weak homotopy equivalences of bisimplicial sets.*

Lemma 1 is an immediate consequence of the following stronger assertion:

**Lemma 2.** *For each integer* $m \geq 0$*, the maps* $\mathrm{N}_m(\mathcal{E}^\circ_\bullet) \to \mathrm{N}_m(\mathcal{D}^\circ_\bullet)$ *and* $\mathrm{N}_m(\mathcal{E}_\bullet) \to \mathrm{N}_m(\mathcal{D}_\bullet)$ *are weak homotopy equivalences of simplicial sets.*

For the remainder of this lecture, let us fix $m \geq 0$. We will prove that the natural map $\mathrm{N}_m(\mathcal{E}_\bullet) \to \mathrm{N}_m(\mathcal{D}_\bullet)$ is a weak homotopy equivalence. The analogous assertion for the map $\mathrm{N}_m(\mathcal{E}^\circ_\bullet) \to \mathrm{N}_m(\mathcal{D}^\circ_\bullet)$ can be proven by exactly the same argument.

We wish to prove that every homotopy fiber $F$ of the map of topological spaces $|\,\mathrm{N}_m(\mathcal{E}_\bullet)| \to |\,\mathrm{N}_m(\mathcal{D}_\bullet)|$ is weakly contractible: that is, that every map from a sphere $S^{n-1}$ into $F$ is nullhomotopic. Any such map can be represented by a commutative diagram of topological spaces $\widehat{\sigma}$:

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\hat{f}} & |\,\mathrm{N}_m(\mathcal{E}_\bullet)| \\ \downarrow & & \downarrow \\ D^n & \xrightarrow{\hat{g}} & |\,\mathrm{N}_m(\mathcal{D}_\bullet)|. \end{array}$$

To prove that this map is nullhomotopic, it will suffice to show that $\hat{f}$ and $\hat{g}$ can be extended to a compatible pair of maps

$$\hat{F} : (S^{n-1} \times [0,1]) \amalg_{S^{n-1} \times \{1\}} D^n \to |\,\mathrm{N}_m(\mathcal{E}_\bullet)|$$

$$\hat{G} : D^n \times [0,1] \to |\,\mathrm{N}_m(\mathcal{D}_\bullet)|.$$

Note that the simplicial set $\mathrm{N}_m(\mathcal{D}_\bullet)$ is not a Kan complex, so that the homotopy class of $\widehat{f}$ cannot necessarily be represented by a map of simplicial sets $\partial\,\Delta^n \to \mathrm{N}_m(\mathcal{D}_\bullet)$. However, such a representation always exists after passing to an iterated subdivision of $\partial\,\Delta^{n-1}$. Applying the same reasoning to $\widehat{g}$, we may assume without loss of generality that $\widehat{\sigma}$ is obtained from a commutative diagram of simplicial sets $\sigma$ :

$$\begin{array}{ccc} A & \xrightarrow{f} & |\,\mathrm{N}_m(\mathcal{E}_\bullet)| \\ \downarrow & & \downarrow \\ B & \xrightarrow{g} & |\,\mathrm{N}_m(\mathcal{D}_\bullet)|. \end{array}$$

where $A$ and $B$ are finite nonsingular simplicial sets whose geometric realizations are homeomorphic to $S^{n-1}$ and $D^n$, respectively. It will therefore suffice to prove the following:

**Lemma 3.** *Let $m \geq 0$ be a nonnegative integer, let $B$ be a finite nonsingular simplicial set, and let $A \subseteq B$, and suppose we are given a commutative diagram $\sigma$ :*

$$\begin{array}{ccc} A & \xrightarrow{f} & |\mathrm{N}_m(\mathcal{E}_\bullet)| \\ \downarrow & & \downarrow \\ B & \xrightarrow{g} & |\mathrm{N}_m(\mathcal{D}_\bullet)|. \end{array}$$

*Then we can extend $\sigma$ to a commutative diagram*

$$\begin{array}{ccccc} A & \longrightarrow & \overline{A} & \xrightarrow{F} & |\mathrm{N}_m(\mathcal{E}_\bullet)| \\ \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & \overline{B} & \xrightarrow{G} & |\mathrm{N}_m(\mathcal{D}_\bullet)|. \end{array}$$

*for which there are homeomorphisms*

$$|\overline{B}| \simeq |B| \times [0,1]$$

$$|\overline{A}| \simeq (|A| \times [0,1]) \amalg_{A \times \{1\}} (B \times \{1\}),$$

*under which the left horizontal maps correspond to triangulations of the inclusions*

$$|B| \times \{0\} \hookrightarrow |B| \times [0,1]$$

$$|\overline{A}| \times \{0\} \hookrightarrow (|A| \times [0,1]) \amalg_{A \times \{1\}} (B \times \{1\}).$$

Let us begin with the construction of the map $G$. Unwinding the definitions (using the fact that fibrations between polyhedra can be tested "simplex-wise"; see Lecture 8), we see that $g$ classifies a diagram of finite polyhedra

$$E_0 \to E_1 \to \cdots \to E_m \to |B|$$

where each of the maps $E_i \to E_j$ is cell-like and each of the maps $E_i \to |B|$ is a fibration. Let $E_i^A \subseteq E_i$ denote the inverse image of the subcomplex $|A| \subseteq |B|$. Then the map $F$ determines a strong triangulation of each $E_i^A$, represented by a PL homeomorphism $E_i^A \simeq |X_i|$ for some finite nonsingular simplicial set $X_i$.

**Definition 4.** Let $K$ be a finite polyhedron equipped with strong triangulations $\tau$ and $\tau'$ (as defined in the previous lecture), represented by PL homeomorphisms $K \simeq |Z|$ and $K \simeq |Z'|$. We will say that $\tau$ is a *refinement* of $\tau'$ if the following conditions are satisfied:

- For every nondegenerate simplex $\sigma$ of $Z'$, the image of $|\sigma|$ under the composite map $|\sigma| \subseteq |Z'| \simeq K \simeq |Z|$ is a subcomplex of $|Z|$: that is, it can be identified with the geometric realization of a simplicial subset $Z_\sigma \subseteq Z$.
- The homeomorphism $|Z_\sigma| \simeq |\sigma|$ is linear when restricted to each simplex of $Z_\sigma$.

**Warning 5.** The relation of refinement on strong triangulations is transitive but not antireflexive: it is possible for two strong triangulations $\tau$ and $\tau'$ to refine each other without being equivalent.

We will invoke without proof the following (hopefully plausible) fact from the theory of PL topology:

**Proposition 6.** *In the situation above, there exist strong triangulations of the polyhedra $E_i$ and $L = |B|$, represented by PL homeomorphisms $E_i \simeq |Y_i|$ and $L \simeq |B'|$, with the following properties:*

(*a*) *The diagram $E_0 \to E_1 \to \cdots \to E_m \to L$ can be obtained as the geometric realization of a diagram of simplicial sets $Y_0 \to \cdots \to Y_m \to B'$.*

(b) *Let $K = |A| \subseteq L$. Then the image of $K$ under the homeomorphism $L \simeq |B'|$ is a subcomplex (that is, it can be identified with the geometric realization of a simplicial subset $A' \subseteq B'$). Moreover, the strong triangulation of $K$ determined by the PL homeomorphism $K \simeq |A'|$ refines the strong triangulation determined by the homeomorphism $K \simeq |A|$.*

(c) *It follows from (b) that for $0 \leq i \leq m$, the image of $E_i^A$ in $|Y_i|$ can be identified with the geometric realization of a subcomplex $X_i' \subseteq Y_i$. We require that the strong triangulation determined by the PL homeomorphism $E_i^A \simeq |X_i'|$ refines the strong triangulation determined by the PL homeomorphism $E_i^A \simeq |X_i|$.*

Here $B'$ is a finite simplicial set whose geometric realization is homeomorphic to $|B|$. However, $B$ and $B'$ need not be isomorphic as simplicial sets. Nevertheless, we have the following:

**Proposition 7.** *Let $P$ be a finite polyhedron equipped with strong triangulations $\tau$ and $\tau'$, represented by PL homeomorphisms $P \simeq |Y|$ and $P \simeq |Y'|$. Assume that $\tau'$ refines $\tau$. Then there exist a strong triangulation $\overline{\tau}$ of $P \times [0,1]$, represented by a PL homeomorphism $\beta : P \times [0,1] \simeq |\overline{Y}|$, with the following properties:*

(i) *The maps*

$$|Y| \simeq P \times \{0\} \hookrightarrow P \times [0,1] \simeq |\overline{Y}|$$

$$|Y'| \simeq P \times \{1\} \hookrightarrow P \times [0,1] \simeq |\overline{Y}|$$

*are induced by inclusions of simplicial sets $i : Y \hookrightarrow \overline{Y}$ and $j : Y' \hookrightarrow \overline{Y}$.*

(ii) *For every nondegenerate simplex $\sigma$ of $Y$, the image $\beta(|\sigma| \times [0,1])$ can be identified with the geometric realization of a simplicial subset $\overline{Y}_\sigma \subseteq \overline{Y}$, and the resulting PL homeomorphism $|\overline{Y}_\sigma| \simeq |\sigma| \times [0,1]$ is linear on each simplex.*

*Proof.* We proceed by induction on the number of nondegenerate simplices of $Y$. If $Y$ is empty there is nothing to prove; otherwise, we may assume that $Y$ is obtained from a simplicial subset $Y_0 \subsetneq Y$ by adding a single nondegenerate $k$-simplex. Let $P_0 \subseteq P$ be the image of $|Y_0|$. Since $\tau'$ refines $\tau$, the homeomorphism $P \simeq |Y'|$ carries $_0$ to the geometric realization of a simplicial subset $Y_0' \subseteq Y'$. By virtue of our inductive hypothesis, we may assume that there exists a strong triangulation $\beta_0 : P_0 \times [0,1] \simeq |\overline{Y}_0|$ satisfying the analogues of conditions (i) and (ii). We claim that it is possible to extend this to a strong triangulation of $P \times [0,1]$ satisfying the same conditions. The problem of finding this extension is "local": that is, we may assume that $Y = \Delta^k$ and that $Y_0 = \partial \Delta^k$. Set $Z = Y \amalg_{Y_0} \overline{Y}_0 \amalg_{Y_0'} Y'$. Then $\beta_0$, together with $\tau$ and $\tau'$, determine a PL homeomorphism $\beta_1 : |Z| \simeq \partial(P \times [0,1])$ which is linear on each simplex. Let $\overline{Y}$ be the cone on $Z$. For any point $v$ in the interior of $P$, there is a unique PL homeomorphism $\beta : |\overline{Y} \simeq P \times [0,1]$ which extends $\beta_1$, is linear on each simplex, and carries the cone point of $\overline{Y}$ to the point $(v, \frac{1}{2}) \in P \times [0,1]$. It is easy to see that $\beta$ has the desired properties. □

**Remark 8.** The proof of Proposition 7 gives an explicit *construction* of the strong triangulation $\beta : |\overline{Y}| \simeq P \times [0,1]$. However, the construction requires making some auxiliary choices: namely, for each nondegenerate simplex $\sigma$ of $Y$, we need to choose a point $v_\sigma \in P$ belonging to the interior of the image of $|\sigma|$.

Let $Q$ be another finite polyhedron equipped with strong triangulations $Q \simeq |Z|$ and $Q \simeq |Z'|$ satisfying the hypotheses of Proposition 7, and let $q : Q \to P$ be a PL map which is induced by maps of simplicial sets $f : Z \to Y$ and $f' : Z' \to Y'$. Suppose that, for each nondegenerate simplex $\sigma$ of $Z$, we choose a point $u_\sigma \in Q$ belonging to the interior of the image of $|\sigma|$. We can then apply the proof of Proposition 7 to obtain a triangulation $Q \times [0,1] \simeq |\overline{Z}|$. If the map $q$ satisfies $q(u_\sigma) = v_{f(\sigma)}$ for each $\sigma$, then the induced map $Q \times [0,1] \to P \times [0,1]$ arises from a map of simplicial sets $\overline{Z} \to \overline{Y}$ which is compatible with the maps $f$ and $f'$ above. Note that the condition $q(u_\sigma) = v_{f(\sigma)}$ can always be achieved by choosing the points $u_\sigma$ appropriately.

Applying Proposition 7 to the triangulations $|B| \simeq L$ and $|B'| \simeq L$, we obtain a triangulation $L \times [0,1] \simeq |\overline{B}|$. We then have cell-like maps of finite polyhedra

$$E_0 \times [0,1] \to E_1 \times [0,1] \to \cdots \to E_m \times [0,1]$$

which are fibered over $|\overline{B}|$. Choosing auxiliary embeddings into $|\overline{B}| \times \mathbb{R}^\infty$, we may assume that this diagram is classified by a map $G : \overline{B} \to \mathrm{N}_m(\mathcal{D}_\bullet)$.

By construction, the image of $K \times [0,1]$ under the PL homeomorphism $L \times [0,1] \simeq |\overline{B}|$ can be identified with the geometric realization of a simplicial subset $\overline{A}_0 \subseteq \overline{B}$. Let $\overline{A}$ denote the union of $\overline{A}_0$ with the image of the inclusion $B' \hookrightarrow \overline{B}$, so that

$$|\overline{A}| \simeq (K \times [0,1]) \amalg_{K \times \{1\}} L.$$

To complete the proof, it will suffice to show that $G|_{\overline{A}}$ can be lifted to a map $F : \overline{A} \to |\mathrm{N}_m(\mathcal{E}_\bullet)|$ which extends the map $f$. To find this extension, we must find strong triangulations of the fiber products

$$|\overline{A}| \times_{|\overline{B}|} (E_i \times [0,1])$$

for which the maps

$$|\overline{A}| \times_{|\overline{B}|} (E_0 \times [0,1]) \to \cdots \to |\overline{A}| \times_{|\overline{B}|} (E_m \times [0,1]) \to |\overline{A}|$$

are simplicial and which restrict to the strong triangulations $E_i^A \simeq |X_i|$ determined by $f$. In fact, we will choose these strong triangulations so that they are also compatible with the strong triangulations $E_i \times \{1\} \simeq |Y_i|$ required by Proposition 6. These triangulations are obtained by applying Proposition 7 to obtain strong triangulations of $E_i^A \times [0,1]$ which are compatible with the given strong triangulations $E_i^A \times \{0\} \simeq |X_i|$ and $E_i^A \times \{1\} \simeq |X_i'|$, iteratively applying Remark 8 to guarantee that the maps

$$E_0^A \times [0,1] \to \cdots \to E_m^A \to [0,1] \to |\overline{A}_0|$$

are compatible with the chosen triangulations.

## References

[1] Waldhausen, F., B. Jahren and J. Rognes. *Spaces of PL Manifolds and Categories of Simple Maps.*