Combinatorics I (Lecture 36)

February 18, 2015

Our goal in this lecture (and the next) is to carry out the combinatorial steps in the proof of the main theorem of Part III. We begin by recalling some notation from the previous lecture.

Notation 1. For each $k \geq 0$, we let $\mathcal{D}_k$ denote the category whose objects are finite polyhedra $E \subseteq \Delta^k \times \mathbb{R}^\infty$ for which the projection $E \to \Delta^k$ is a fibration, and whose morphisms are cell-like maps $E \to E'$ which commute with the projection to Δ^k. We let $\mathcal{D}_k^\circ$ denote the full subcategory of $\mathcal{D}_k$ spanned by those objects for which the map $E \to \Delta^k$ is a fiber bundle (in the PL category); that is, for which there is a PL homeomorphism $E \simeq \Delta^k \times K$ for some finite polyhedron K.

The categories $\mathcal{D}_k$ and $\mathcal{D}_k^\circ$ depend functorially on the linearly ordered set $[k] = \{0 < \cdots < k\}$, and therefore determine simplicial categories $\mathcal{D}_\bullet$ and $\mathcal{D}_\bullet^\circ$.

Warning 2. We have modified the meaning of the notation $\mathcal{D}_\bullet^\circ$ slightly from the previous lecture: namely, we have replaced our previous $\mathcal{D}_\bullet^\circ$ with its essential image in $\mathcal{D}_\bullet$. This modification does not change the homotopy type, but will be slightly more convenient to work with in this lecture.

Recall that our goal is to prove the following:

Proposition 3. *The canonical maps*

$$\mathrm{N}_\bullet(\mathcal{D}_\bullet^\circ) \overset{\alpha}{\hookrightarrow} \mathrm{N}_\bullet(\mathcal{D}_\bullet) \overset{\beta}{\hookleftarrow} \mathrm{N}_0(\mathcal{D}_\bullet) = \mathcal{M}$$

are weak homotopy equivalences (of bisimplicial sets).

Let $\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}}$ denote the category whose objects are finite nonsingular simplicial sets and whose morphisms are cell-like maps. Let $\mathrm{N}_\bullet^{\mathrm{op}}(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}})$ denote the nerve of the opposite category (regarded as a simplicial set). Recall that in Lecture 10, we constructed a weak homotopy equivalence $\alpha' : \mathrm{N}_\bullet^{\mathrm{op}}(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}}) \to \mathcal{M}$. Note that the category $\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}}$ is equipped with a forgetful functor $\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}} \to \mathcal{D}_0^\circ$, which assigns to each finite nonsingular simplicial set X the polyhedron $|X|$ (which we can regard as an object of $\mathcal{D}_0^\circ$ by choosing an arbitrary PL embedding of $|X|$ into $\mathbb{R}^\infty$). This determines a map of bisimplicial sets

$$\beta' : \mathrm{N}_\bullet^{\mathrm{op}}(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}}) \to \mathrm{N}_\bullet^{\mathrm{op}}(\mathcal{D}_\bullet^\circ),$$

whose domain is constant in one direction.

The first step is to establish the following:

Lemma 4. *The diagram of bisimplicial sets*

$$\begin{array}{ccc} \mathrm{N}_\bullet^{\mathrm{op}}(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}}) & \xrightarrow{\alpha'} & \mathcal{M} \\ \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \beta} \\ \mathrm{N}_\bullet^{\mathrm{op}}(\mathcal{D}_\bullet^\circ) & \xrightarrow{\alpha} & \mathrm{N}_\bullet^{\mathrm{op}}(\mathcal{D}_\bullet) \end{array}$$

commutes up to homotopy. Consequently, to prove that β is a weak homotopy equivalence, it will suffice to show that α and β' are weak homotopy equivalences (since the upper horizontal map is a weak homotopy equivalence; see Lecture 10).

Let us prove Lemma 4. Let $Z_\bullet$ denote the simplicial set given by the diagonal of $\mathrm{N}^{\mathrm{op}}_\bullet(\mathcal{D}_\bullet)$. Unwinding the definitions, we see that an n-simplex of $Z_\bullet$ is given by a sequence of finite polyhedra $X_0, \ldots, X_n \subseteq \Delta^n \times \mathbb{R}^\infty$ for which the projection maps $X_i \to \Delta^n$ are fibrations, together with cell-like PL maps

$$X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n.$$

We will denote such an n-simplex simply by $X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n$.

An n-simplex of $\mathrm{N}^{\mathrm{op}}_\bullet(\mathcal{S}\mathrm{et}^{\mathrm{ns}}_\Delta)$ is a chain of cell-like maps

$$X_0 \leftarrow \cdots \leftarrow X_n$$

of nonsingular simplicial sets. If X is a nonsingular simplicial set, we let $\Sigma(X)$ denote the poset of nondegenerate simplices of X. Given a chain

$$X_0 \leftarrow \cdots \leftarrow X_n$$

as above, we let $\Sigma(\vec{X})$ denote the disjoint union

$$\Sigma(X_0) \amalg \Sigma(X_1) \amalg \cdots \amalg \Sigma(X_n),$$

partially ordered so that a simplex $\sigma \subseteq X_i$ is $\leq$ a simplex $\tau \subseteq X_j$ if and only if $i \leq j$ and σ is contained in the image of τ. Recalling the definition of the map α', we note that the composition $\beta \circ \alpha'$ carries a chain

$$X_0 \leftarrow \cdots \leftarrow X_n$$

to the n-simplex of Z given by

$$|\,\mathrm{N}(\Sigma(\vec{X}))| \xleftarrow{\mathrm{id}} |\,\mathrm{N}(\Sigma(\vec{X})| \leftarrow \cdots \leftarrow |\,\mathrm{N}(\Sigma(\vec{X}))|.$$

On the other hand, the composition $\beta \circ \alpha'$ carries the same chain to the diagram

$$|X_0 \times \Delta^n| \leftarrow |X_1 \times \Delta^n| \leftarrow \cdots \leftarrow |X_n \times \Delta^n|.$$

For $X_0 \leftarrow \cdots \leftarrow X_n$ as above and $0 \leq i \leq n$, let $\vec{X}^{\geq i}$ and $\vec{X}^{=i}$ denote the sequences given by

$$\vec{X}^{\geq i}_j = \begin{cases} X_i & \text{if } j \leq i \\ X_j & \text{if } j \geq i. \end{cases}$$

$$\vec{X}^{=i}_j = X_i.$$

Let Sd denote the subdivision functor from $\mathcal{S}\mathrm{et}^{\mathrm{ns}}_\Delta$ to itself. Since the "last vertex map" $\mathrm{Sd}(X) \to X$ is cell-like for every nonsingular simplicial set X, Sd induces a map from $\mathrm{N}^{\mathrm{op}}_\bullet(\mathcal{S}\mathrm{et}^{\mathrm{ns}}_\Delta)$ to itself which is homotopic to the identity, so that $\beta \circ \alpha'$ is homotopic to the map $\beta \circ \alpha' \circ \mathrm{Sd}$ which carries a chain $X_0 \leftarrow \cdots \leftarrow X_n$ to the diagram

$$|\,\mathrm{N}(\Sigma(\vec{X}^{=0}))| \leftarrow |\,\mathrm{N}(\vec{X}^{=1})| \leftarrow \cdots \leftarrow |\,\mathrm{N}(\vec{X}^{=n})|.$$

We define a map of simplicial sets $\gamma : \mathrm{N}^{\mathrm{op}}_\bullet(\mathcal{S}\mathrm{et}^{\mathrm{ns}}_\Delta) \to Z_\bullet$ which carries an n-simplex $X_0 \leftarrow \cdots \leftarrow X_n$ to the diagram

$$|\,\mathrm{N}(\Sigma(\vec{X}^{\geq 0}))| \leftarrow |\,\mathrm{N}(\vec{X}^{\geq 1})| \leftarrow \cdots \leftarrow |\,\mathrm{N}(\vec{X}^{\geq n})|$$

(together with an appropriately chosen PL embedding of each $|\,\mathrm{N}(\Sigma(\vec{X}^{\geq i}))|$ into $\Delta^n \times \mathbb{R}^\infty$). We claim that γ is well-defined, and that the natural nondecreasing maps

$$\Sigma(\vec{X}) \leftarrow \Sigma(\vec{X}^{\geq i}) \to \Sigma(\vec{X}^{=i})$$

determine homotopies from γ to $\alpha \circ \beta'$ and $\beta \circ \alpha' \circ \mathrm{Sd}$, respectively. To prove this, it suffices to verify the following (for $X_0 \leftarrow \cdots \leftarrow X_n$ as above):

(*a*) For $0 \leq i < n$, the natural map of posets $\Sigma(\vec{X}^{\geq i+1}) \to \Sigma(\vec{X}^{\geq i})$ induces a cell-like map of nerves.

(*b*) For $0 \leq i \leq n$, the natural map of posets $\Sigma(\vec{X}^{\geq i}) \to \Sigma(\vec{X})$ induces a cell-like map of nerves.

(*c*) For $0 \leq i \leq n$, the natural map of posets $\Sigma(\vec{X}^{\geq 0}) \to \Sigma(\vec{X}^{=i})$ induces a cell-like map of nerves.

Note that $\vec{X} = \vec{X}^{\geq 0}$, so that assertion (*b*) follows from iterated applications of (*a*). To prove (*a*) and (*c*), we note that the poset maps in question are Cartesian fibrations. It will therefore suffice to show that they have weakly contractible fibers (by the first Proposition from Lecture 10). This follows from the fact that the poset maps $\Sigma(X_j) \to \Sigma(X_i)$ have weakly contractible fibers for $j \geq i$, by virtue of our assumption that the maps $X_j \to X_i$ are cell-like. This completes the proof of Lemma 4.

Let us now return to the proof of Proposition 3.

Definition 5. Let K be a polyhedron. A *strong triangulation* of K is an equivalence class of PL homeomorphisms $h : K \simeq |X|$, where X is a finite nonsingular simplicial sets. Two such homeomorphisms $h : K \simeq |X|$ and $h' : K \simeq |X'|$ are considered equivalent if the composite homeomorphism $h' \circ h^{-1} : |X| \to |X'|$ arises from a map of simplicial sets $X \to X'$ (necessarily unique).

If $f : K \to L$ is a map of finite polyhedra equipped with strong triangulations given by PL homeomorphisms $h : K \simeq |X|$ and $h' : L \simeq |Y|$, then we say that f is *compatible with strong triangulations* if the composite map $|X| \xrightarrow{h^{-1}} K \xrightarrow{f} L \xrightarrow{h'} |Y|$ arises from a (necessarily unique) map of simplicial sets $X \to Y$. Note that in this case, f is automatically PL.

It will be convenient to introduce a variation on Notation **??**:

Variant 6. For each $k \geq 0$, we let $\mathcal{E}_k$ denote the category whose objects are finite polyhedra $E \subseteq \Delta^k \times \mathbb{R}^\infty$ equipped with a strong triangulation for which the projection $p : E \to \Delta^k$ is a fibration, together with a strong triangulation of E which is compatible with p. The morphisms in $\mathcal{E}_k$ are cell-like maps $E \to E'$ which are compatible with the strong triangulations and commute with the projection to Δ^k. We let $\mathcal{E}_k^\circ$ denote the full subcategory of $\mathcal{E}_k$ spanned by those objects for which the map $E \to \Delta^k$ is a fiber bundle (in the PL category).

Note that the map $\beta' : \mathrm{N}_\bullet(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}}) \to \mathrm{N}_\bullet(\mathcal{D}_\bullet^\circ)$ factors through $\mathrm{N}_\bullet(\mathcal{E}_\bullet^\circ)$, so that we have a commutative diagram of bisimplicial sets

$$\begin{array}{ccc} \mathrm{N}_\bullet(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}}) & & \\ \downarrow & \searrow & \\ \mathrm{N}_\bullet(\mathcal{E}_\bullet^\circ) & \longrightarrow & \mathrm{N}_\bullet(\mathcal{E}_\bullet) \\ \downarrow & & \downarrow \\ \mathrm{N}_\bullet(\mathcal{D}_\bullet^\circ) & \xrightarrow{\alpha} & \mathrm{N}_\bullet(\mathcal{D}_\bullet). \end{array}$$

Using this diagram and Lemma 4, we see that Proposition 3 is a consequence of the following pair of results:

Lemma 7. *The vertical maps* $\mathrm{N}_\bullet(\mathcal{E}_\bullet^\circ) \to \mathrm{N}_\bullet(\mathcal{D}_\bullet^\circ)$ *and* $\mathrm{N}_\bullet(\mathcal{E}_\bullet) \to \mathrm{N}_\bullet(\mathcal{D}_\bullet)$ *are weak homotopy equivalences of bisimplicial sets.*

Lemma 8. *The natural maps*

$$\mathrm{N}_\bullet(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}}) \to \mathrm{N}_\bullet(\mathcal{E}_\bullet^\circ) \to \mathrm{N}_\bullet(\mathcal{E}_\bullet)$$

are weak homotopy equivalences of bisimplicial sets (the first of which is constant in one direction).

Lemma 7 articulates the idea that every polyhedron has a "contractible" space of triangulations; we will prove it in the next lecture. We will deduce Lemma 8 from the following more precise result:

Lemma 9. *For each integer $k \geq 0$, the natural maps*

$$\mathrm{N}_\bullet(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}}) \to \mathrm{N}_\bullet(\mathcal{E}_k^\circ) \to \mathrm{N}_\bullet(\mathcal{E}_k)$$

are weak homotopy equivalences.

Remark 10. When $k = 0$, Lemma 9 is a tautology: the functors $\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}} \to \mathcal{E}_k^\circ \subseteq \mathcal{E}_k$ are equivalences of categories.

We will prove that the natural map $\delta : \mathrm{N}_\bullet(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}}) \to \mathrm{N}(\mathcal{E}_k)$ is a weak homotopy equivalence; the analogous statement for $\mathrm{N}(\mathcal{E}_k^\circ)$ can be proven in the same way. Let us identify $\mathcal{E}_k$ with the category whose objects are finite nonsingular simplicial sets Y with a map $Y \to \Delta^k$ whose geometric realization is a fibration, and whose morphisms are cell-like maps of simplicial sets which commute with the projection to Δ^k. Given such a simplicial set Y, we let $\Sigma(Y)$ denote the poset of nondegenerate simplices of Y, and we let $\Sigma(Y)^{\mathrm{full}}$ denote the subset of $\Sigma(Y)$ consisting of those simplices σ for which the composite map $\sigma \subseteq Y \to \Delta^k$ is surjective. The construction $Y \mapsto \mathrm{N}(\Sigma(Y)^{\mathrm{full}})$ determines a functor ϵ from $\mathcal{E}_k$ to the category $\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}}$. We will deduce Lemma 9 from the following:

Lemma 11. *For $Y \in \mathcal{E}_k$, there is a canonical cell-like map $\theta : \mathrm{N}(\Sigma(Y)^{\mathrm{full}}) \times \Delta^k \to Y$ which commutes with the projection to Δ^k and depends functorially on Y.*

Proof of Lemma **??**. Lemma 11 produces a natural transformation from the composite functor

$$\mathcal{E}_k \xrightarrow{\epsilon} \mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}} \xrightarrow{\delta} \mathcal{E}_k$$

to the identity, so that ϵ is a right homotopy inverse to δ. Applying Lemma 11 to a product $X \times \Delta^k$, we obtain a natural cell-like map

$$\mathrm{N}(\Sigma(X \times \Delta^k)^{\mathrm{full}}) \times \Delta^k \to X \times \Delta^k.$$

Passing to the fiber over any vertex of Δ^k, we obtain a natural cell-like map

$$\mathrm{N}(\Sigma(X \times \Delta^k)^{\mathrm{full}}) \to X,$$

which gives a natural transformation from the composition

$$\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}} \xrightarrow{\delta} \mathcal{E}_k \xrightarrow{\epsilon} \mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}}$$

to the identity, so that ϵ is also a left homotopy inverse to δ. □

We now prove Lemma 11.

Construction 12. Let Y be a simplicial set equipped with a map $Y \to \Delta^k$. We can identify the n-simplices of $\mathrm{N}(\Sigma(Y)^{\mathrm{full}}) \times \Delta^k$ with pairs

$$(\sigma_0 \subseteq \cdots \subseteq \sigma_n) \qquad 0 \leq i_0 \leq \cdots \leq i_n \leq k$$

where each σ_i is a nondegenerate simplex of Y which surjects onto Δ^k. For $0 \leq i \leq k$, let $v_i(\sigma_j)$ denote the "last" vertex of σ_j which lies over the ith vertex of Δ^k. There is a unique map $\Delta^n \to \sigma_n$ which carries the ith vertex of Δ^n to $v_{i_j}(\sigma_j)$, and the composite map

$$\Delta^n \to \sigma_n \hookrightarrow Y$$

can be regarded as an n-simplex of Y. This construction depends functorially on $[n]$, and therefore induces a map of simplicial sets

$$\theta : \mathrm{N}(\Sigma(Y)^{\mathrm{full}}) \times \Delta^k \to Y.$$

Remark 13. In the special case $k = 0$, Construction 12 reproduces the "last vertex map" $\mathrm{Sd}(Y) \to Y$. We may therefore view Construction 12 as a "relative version" of the last vertex map.

Construction 12 is functorial in Y. Lemma **??** is therefore a consequence of the following:

Lemma 14. *Let Y be a finite nonsingular simplicial set equipped with a map $Y \to \Delta^k$ whose geometric realization is a fibration. Then the map*

$$\theta : \mathrm{N}(\Sigma(Y)^{\mathrm{full}}) \times \Delta^k \to Y$$

of Construction 12 is cell-like.

Remark 15. In the special case $k = 0$, Lemma 14 was proven in Lecture 10.

Proof of Lemma 14. Note that we can identify $\mathrm{N}(\Sigma(Y)^{\mathrm{full}}) \times \Delta^k$ with the nerve of the poset $\Sigma(Y)^{\mathrm{full}} \times [k]$. Passing to the subdivision, θ induces a Cartesian fibration of posets

$$q : \mathrm{Chain}(\Sigma(Y)^{\mathrm{full}} \times [k]) \to \Sigma(Y).$$

Using the criterion of Lecture 9, we are reduced to showing that the fibers of this map are weakly contractible. Let us therefore fix a nondegenerate simplex τ of Y, let $\rho \simeq \Delta^{k'}$ be its image in Δ^k, and let $\Sigma(Y)^{\rho}$ denote the subset of $\Sigma(Y)$ consisting of those simplices whose image in Δ^k is equal to ρ. The construction $\sigma \mapsto \sigma \times_{\Delta^k} \rho$ determines a Cartesian fibration of posets $\Sigma(Y)^{\mathrm{full}} \to \Sigma(Y)^{\rho}$, which has contractible fibers by virtue of our assumption that the map $Y \to \Delta^k$ is a Serre fibration (see Lecture 9). The pullback of q

$$\mathrm{Chain}(\Sigma(Y)^{\mathrm{full}} \times [k']) \to \Sigma(Y) \times_{\Delta^k} \rho$$

factors as a composition

$$\mathrm{Chain}(\Sigma(Y)^{\mathrm{full}} \times [k']) \to \mathrm{Chain}(\Sigma(Y)^{\rho} \times [k']) \to \Sigma(Y) \times_{\Delta^k} \rho$$

where the second map is given by applying Construction 12 to the projection map $Y \times_{\Delta^k} \rho \to \rho$. Consequently, to prove that the fiber $q^{-1}\{\tau\}$ is weakly contractible, we can replace Y by $Y \times_{\Delta^k} \rho$ and thereby reduce to the case where $\rho = \Delta^k$: that is, the simplex $\tau \subseteq Y$ is full.

Let $\Sigma(Y)^{\mathrm{init}}$ denote the subset of $\Sigma(Y)$ consisting of those nondegenerate simplices of Y whose image in Δ^k is spanned by the vertices $\{0, 1, 2, \ldots, i\}$ for some $0 \leq i \leq k$. The construction $(\sigma, i) \mapsto \sigma \times_{\Delta^k} \Delta^i$ determines a map of partially ordered sets $r : \Sigma(Y)^{\mathrm{full}} \times [k] \to \Sigma(Y)^{\mathrm{init}}$, and the map q is given by the composition

$$\mathrm{Chain}(\Sigma(Y)^{\mathrm{full}} \times [k]) \stackrel{\mathrm{Chain}(r)}{\to} \mathrm{Chain}(\Sigma(Y)^{\mathrm{init}}) \subseteq \mathrm{Chain}(\Sigma(Y)) \stackrel{q'}{\to} \Sigma(Y),$$

where q' is the last vertex map. We will prove:

- (a) The map $\mathrm{Chain}(r)$ is a Cartesian fibration with weakly contractible fibers.
- (b) The map $q'|_{\mathrm{Chain}(\Sigma(Y)^{\mathrm{init}})}$ has weakly contractible fiber over any $\tau \in \Sigma(Y)^{\mathrm{full}}$.

It will then follow from Quillen's theorem A that the map q has weakly contractible fiber over any $\tau \in \Sigma(Y)^{\mathrm{full}}$.

In Lecture 10, we proved that the map q' has weakly contractible fibers. The proof proceeded by constructing an explicit contracting homotopy for each fiber $q'^{-1}\{\tau\}$. To prove (b), it suffices to observe that when $\tau \in \Sigma(Y)^{\mathrm{full}}$, these explicit contracting homotopies preserve the intersection of $q'^{-1}\{\tau\}$ with $\mathrm{Chain}(\Sigma(Y)^{\mathrm{init}})$ (we leave the verification of this to the reader).

Let us prove (a). It is obvious that $\mathrm{Chain}(r)$ is a Cartesian fibration; we must show that it has contractible fibers. Fix an element of $\mathrm{Chain}(\Sigma(Y)^{\mathrm{init}})$, given by a chain of simplices $\tau_0 \subseteq \cdots \subseteq \tau_m$ in $\Sigma(Y)^{\mathrm{init}}$. Let $P(\tau_0, \ldots, \tau_m)$ denote the inverse image of this chain under $\mathrm{Chain}(r)$; we wish to show that the poset $P(\tau_0, \ldots, \tau_m)$ is weakly contractible. We first treat the case $m = 0$. Let $\Delta^i \subseteq \Delta^k$ be the image of τ_0.

Then $P(\tau_0)$ can be identified with $\mathrm{Chain}(Q)$, where $Q \subseteq \Sigma(Y)^{\mathrm{full}}$ is the set of those simplices σ for which $\sigma \times_{\Delta^k} \Delta^i = \tau_0$. This poset is weakly contractible by virtue of our assumption that the map $Y \to \Delta^k$ is a Serre fibration (see Lecture 9).

Now suppose that $m > 0$. We may assume by induction that the partially ordered set $P(\tau_1, \ldots, \tau_m)$ is weakly contractible. Note that there is a Cartesian fibration of posets

$$f : P(\tau_0, \ldots, \tau_m) \to P(\tau_1, \ldots, \tau_m).$$

Using Quillen's Theorem A, we are reduced to proving that f has contractible fibers. Unwinding the definitions, we see that every such fiber has the form $\mathrm{Chain}(R)$, where R is the subset of $\Sigma(\sigma)^{\mathrm{full}}$ spanned by those simplices σ' satisfying $\sigma' \times_{\Delta^k} \Delta^i = \tau_0$, where $\sigma \in \Sigma(Y)^{\mathrm{full}})$ is some simplex which contains τ_0. The weak contractibility of R follows from applying the criterion of Lecture 9 to the composite map $\sigma \subseteq Y \to \Delta^k$, which is a surjective linear map of simplices and is therefore a Serre fibration. $\square$

References

[1] Waldhausen, F., B. Jahren and J. Rognes. *Spaces of PL Manifolds and Categories of Simple Maps.*