The Setup (Lecture 35)

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Let us begin by recalling some of our main characters.

Notation 1. We let $\mathcal{M}$ denote the "classifying space" for simple homotopy types: that is, the Kan complex whose k-simplices are finite polyhedra $E \subseteq \Delta^k \times \mathbb{R}^\infty$ for which the projection map $E \to \Delta^k$ is a fibration.

For each integer $d \geq 0$, we let $\mathcal{M}\mathrm{an}^d$ denote the Kan complex whose k-simplices are pairs (E, ρ), where $E \subseteq \Delta^k \times \mathbb{R}^\infty$ is a finite polyhedron, the projection $E \to \Delta^k$ is a PL fiber bundle whose fibers are PL manifolds of dimension d (possibly with boundary), and ρ is a trivialization of the (relative) tangent microbundle T_{E/Δ^k}. Note that the construction $(E, \rho) \mapsto E$ determines a map of simplicial sets $\theta_d : \mathcal{M}\mathrm{an}^d \to \mathcal{M}$.

Forming the product with $[0, 1]$ (and composing with an embedding $\mathbb{R}^\infty \times [0, 1] \hookrightarrow \mathbb{R}^\infty$) we obtain stabilization maps

$$\mathcal{M} \to \mathcal{M} \to \mathcal{M} \to \cdots$$

$$\mathcal{M}\mathrm{an}^0 \to \mathcal{M}\mathrm{an}^1 \to \mathcal{M}\mathrm{an}^2 \to \cdots$$

which are compatible with the forgetful maps θ_d. We therefore obtain a map of simplicial sets

$$\theta_\infty : \mathcal{M}\mathrm{an}^\infty = \varinjlim_d \mathcal{M}\mathrm{an}^d \to \varinjlim_d \mathcal{M} \simeq \mathcal{M}.$$

Variant 2. For each integer $d \geq 0$, we define simplicial sets A^d, B^d, C^d, and D^d as follows:

- (a) A k-simplex of A^d is a k-simplex (E, ρ) of $\mathcal{M}\mathrm{an}^d$ having the property that each component of each fiber of E has nonempty boundary.
- (b) A k-simplex of A^d is a pair (E, ρ) where $E \subseteq \Delta^k \times \mathbb{R}^\infty$ is a finite polyhedron, the projection $E \to \Delta^k$ is a fiber bundle whose fibers are PL manifolds of dimension d, each component of which has nonempty boundary, and ρ is a PL immersion from E to $\Delta^k \times \mathbb{R}^d$ which commutes with the projection to Δ^k.
- (c) A k-simplex of C^d is a k-simplex (E, ρ) of B^d where the map ρ is an embedding.
- (d) A k-simplex of D^d is a finite polyhedron $E \subseteq \Delta^k \times \mathbb{R}^d$ for which the projection $E \to \Delta^k$ is a fiber bundle whose fibers are PL manifolds of dimension d.

Note that every immersion of a PL d-manifold into $\mathbb{R}^d$ determines a trivialization of its tangent microbundle. We therefore have canonical maps of simplicial sets

$$D^d \leftarrow C^d \subseteq B^d \to A^d \subseteq \mathcal{M}\mathrm{an}^d.$$

Moreover, there are evident stabilization maps for each of these simplicial sets (which increase d by 1), given by forming the product with $[0, 1]$. We make the following observations:

- The map $\varinjlim A^d \to \varinjlim \mathcal{M}\mathrm{an}^d$ is an isomorphism of simplicial sets. This follows from the observation that each of the stabilization maps $\mathcal{M}\mathrm{an}^d \to \mathcal{M}\mathrm{an}^{d+1}$ factors through $A^{d+1} \subseteq \mathcal{M}\mathrm{an}^{d+1}$ (since a product $M \times [0, 1]$ always has nonempty boundary).

- Each of the maps $B^d \to A^d$ is a homotopy equivalence of Kan complexes. This follows from the work of Haefliger-Poenaru on piecewise linear immersion theory ([1]).

- Each of the maps $C^d \to D^d$ is a trivial Kan fibration (this is trivial: it essentially amounts to the observation that the space of embeddings from a PL manifold M into $\mathbb{R}^\infty$ is contractible).

- The inclusion $\varinjlim B^d \hookrightarrow \varinjlim C^d$ is a homotopy equivalence. This follows from elementary general position arguments. For example, suppose that we wish to prove surjectivity on π_0. Unwinding the definitions, we wish to show that if M is a PL d-manifold equipped with an immersion $\rho : M \to \mathbb{R}^d$, then after replacing M by some product $M \times [0,1]^k$ we can arrange that ρ is isotopic (through immersions) to an embedding. To prove this, we choose $k \gg 0$ and an embedding $e = (e_1, \ldots, e_k) : M \to [0,1]^k$. Using the fact that ρ is an immersion, we deduce that there exists $\epsilon > 0$ for which the map
$$M \times [0,1]^k \to \mathbb{R}^{d+k}$$
$$(x, t_1, \ldots, t_k) \mapsto (\rho(x), e_1(x) + \epsilon t_1, \ldots, e_k(x) + \epsilon t_k)$$
is an embedding, and it is not difficult to check that this embedding is PL isotopic (through immersions) to the $\rho \times \mathrm{id}$.

It follows that we obtain homotopy equivalences
$$\varinjlim D^d \leftarrow \varinjlim C^d \subseteq \varinjlim B^d \to \varinjlim A^d \subseteq \varinjlim \mathcal{M}\mathrm{an}^d .$$
In other words, the direct limit $\mathcal{M}\mathrm{an}^\infty = \varinjlim \mathcal{M}\mathrm{an}^d$ can be identified with a classifying space for embedded PL submanifolds $M \subseteq \mathbb{R}^d$ (stabilized by taking the dimension d to infinity).

Our goal over the next several lectures is to prove the following:

Theorem 3. *The map θ_∞ is a homotopy equivalence of Kan complexes.*

Let us begin by trying to analyze an individual map θ_d. By construction $\mathcal{M}\mathrm{an}^d$ is a classifying space for compact framed PL manifolds of dimension d. Consequently, $\mathcal{M}\mathrm{an}^d$ is homotopy equivalent to a disjoint union
$$\amalg_M \mathrm{BAut}(M)$$
where the disjoint union is taken over all isomorphism classes of compact framed PL manifolds M of dimension d, and $\mathrm{Aut}(M)$ denotes the (simplicial) group of framed PL homeomorphisms of M with itself. Let us make this identification more explicit. In what follows, we will use the term "simplicial category" to mean a simplicial object in the category of categories, and we will use the term "simplicially enriched category" to mean a category enriched over simplicial sets: that is, a simplicial category where the simplicial set of objects is constant.

Notation 4. Fix an integer $d \geq 0$. For each $k \geq 0$, we let $\mathcal{C}_k$ denote the category whose objects are pairs (E, ρ), where $E \subseteq \Delta^k \times \mathbb{R}^\infty$ is a finite polyhedron, the projection $E \to \Delta^k$ is a PL fiber bundle whose fibers are PL manifolds of dimension d (possibly with boundary), and ρ is a trivialization of the (relative) tangent microbundle T_{E/Δ^k}. A morphism from (E, ρ) to (E', ρ') is a PL homeomorphism of E with E', compatible with the projection to Δ^k, which carries ρ to ρ'. We will regard $\mathcal{C}_\bullet$ as a simplicial category. Let $\mathcal{C}^\circ_\bullet$ denote the "underlying" simplicially enriched category, whose objects are the objects of the category $\mathcal{C}_0$ (which we can identify with framed PL d-manifolds, if we ignore the data of an embedding into $\mathbb{R}^\infty$).

It is not difficult to see that the homotopy type of the disjoint union $\amalg_M \mathrm{BAut}(M)$ is modeled by the bisimplicial set $\mathrm{N}_\bullet(\mathcal{C}^\circ_\bullet)$. On the other hand, we have a canonical isomorphism of simplicial sets $\mathcal{M}\mathrm{an}^d \simeq \mathrm{N}_0(\mathcal{C}_\bullet)$. The existence of a homotopy equivalence $\mathcal{M}\mathrm{an}^d \simeq \amalg_M \mathrm{BAut}(M)$ is a consequence of the following:

Proposition 5. *The canonical maps*

$$\mathrm{N}_\bullet(\mathcal{C}^\circ_\bullet) \hookrightarrow \mathrm{N}_\bullet(\mathcal{C}_\bullet) \hookleftarrow \mathrm{N}_0(\mathcal{C}_\bullet) = \mathcal{M}\mathrm{an}^d$$

are weak homotopy equivalences (of bisimplicial sets).

Proof. The first map is a weak homotopy equivalence because for each integer k, the inclusion $\mathcal{C}^\circ_k \hookrightarrow \mathcal{C}_k$ is an equivalence of categories (this follows from the observation that any PL fiber bundle $E \to \Delta^k$ is trivial, because Δ^k is contractible). To show that the second map is a weak homotopy equivalence, it will suffice to show that for each integer n, the degeneracy map

$$\mathrm{N}_0(\mathcal{C}_\bullet) \hookrightarrow \mathrm{N}_n(\mathcal{C}_\bullet)$$

is a homotopy equivalence of Kan complexes. This map has a left inverse q, given by evaluation at any choice of vertex in Δ^n. It now suffices to show that the map $q : \mathrm{N}_n(\mathcal{C}_\bullet) \to \mathrm{N}_0(\mathcal{C}_\bullet)$ is a trivial Kan fibration. This follows from the contractibility of the space of embeddings of a PL manifold M into $\mathbb{R}^\infty$; we leave the details to the reader. □

We now consider a variant of Notation 4.

Notation 6. For each $k \geq 0$, we let $\mathcal{D}_k$ denote the category whose objects are finite polyhedra $E \subseteq \Delta^k \times \mathbb{R}^\infty$ for which the projection $E \to \Delta^k$ is a fibration, and whose morphisms are cell-like maps $E \to E'$ which commute with the projection to Δ^k. We will regard $\mathcal{D}_\bullet$ as a simplicial category. Let $\mathcal{D}^\circ_\bullet$ denote the "underlying" simplicially enriched category. Ignoring the data of the PL embeddings, we can think of $\mathcal{D}^\circ_\bullet$ as the simplicially enriched category whose objects are finite polyhedra K, where $\mathrm{Map}_{\mathcal{D}^\circ_\bullet}(K, K')$ is the simplicial set parametrizing cell-like maps from K to K'.

Note that we have a canonical isomorphism of simplicial sets $\mathcal{M} \simeq \mathrm{N}_0(\mathcal{D}_\bullet)$. We have the following analogue of Proposition 5:

Proposition 7. *The canonical maps*

$$\mathrm{N}_\bullet(\mathcal{D}^\circ_\bullet) \overset{\alpha}{\hookrightarrow} \mathrm{N}_\bullet(\mathcal{D}_\bullet) \overset{\beta}{\hookleftarrow} \mathrm{N}_0(\mathcal{D}_\bullet) = \mathcal{M}$$

are weak homotopy equivalences (of bisimplicial sets).

Unlike Proposition 5, Proposition 7 is not a triviality. The first part of the proof breaks down because the inclusions $\mathcal{D}^\circ_k \hookrightarrow \mathcal{D}$ are not equivalences of categories (a PL fibration $E \to \Delta^k$ need not be a fiber bundle), and the second part of the proof breaks down because cell-like maps need not be invertible. We will give the proof of Proposition 7 in the next lecture. For the moment, let us study its consequences.

For fixed $d \geq 0$, we have a commutative diagram (of bisimplicial sets)

$$\begin{array}{ccccc} \mathrm{N}_\bullet(\mathcal{C}^\circ_\bullet) & \longrightarrow & \mathrm{N}_\bullet(\mathcal{C}_\bullet) & \longleftarrow & \mathcal{M}\mathrm{an}^d \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \theta_d} \\ \mathrm{N}_\bullet(\mathcal{D}^\circ_\bullet) & \longrightarrow & \mathrm{N}_\bullet(\mathcal{D}_\bullet) & \longleftarrow & \mathcal{M}. \end{array}$$

Consequently, we can identify θ_d with the map of bisimplicial sets $\mathrm{N}_\bullet(\mathcal{C}^\circ_\bullet) \to \mathrm{N}_\bullet(\mathcal{D}^\circ_\bullet)$ induced by the forgetful functor $\mathcal{C}^\circ_\bullet \to \mathcal{D}^\circ_\bullet$ (which associates to each framed PL manifold its underlying finite polyhedron).

It follows from general nonsense that the homotopy type of $\mathrm{N}_\bullet(\mathcal{C}^\circ_\bullet)$ can be expressed as an iterated homotopy colimit

$$\mathrm{hocolim}_{K \in \mathcal{D}^\circ_\bullet}(\mathrm{hocolim}_{M \in \mathcal{C}^\circ_\bullet} \mathrm{Map}_{\mathcal{D}^\circ}(M, K)).$$

Let us fix an object $K \in \mathcal{D}^\circ_\bullet$ for the moment. It follows from Proposition 5 that we can identify the nerve $\mathrm{N}_\bullet(\mathcal{C}^\circ_\bullet)$ with the classifying space $\mathcal{M}\mathrm{an}^d$ for framed PL manifolds of dimension d. The homotopy colimit $\mathcal{M}\mathrm{an}^d_K = (\mathrm{hocolim}_{M \in \mathcal{C}^\circ_\bullet} \mathrm{Map}_{\mathcal{D}^\circ}(M, K))$ is equipped with a canonical map

$$\mathcal{M}\mathrm{an}^d_K \to \mathrm{hocolim}_{M \in \mathcal{C}^\circ_\bullet} * \simeq \mathcal{M}\mathrm{an}^d,$$

which is a fibration classified by the functor $M \mapsto \mathrm{Map}^\circ_{\mathcal{D}}(M, K)$. More explicitly, we can identify $\mathcal{M}\mathrm{an}^d_K$ with the simplicial set whose k-simplices are triples (E, ρ, q), where (E, ρ) is a k-simplex of $\mathcal{M}\mathrm{an}^d$ and $q : E \to K$ is a PL map which is cell-like on each fiber of E. The above analysis then gives

$$\mathcal{M}\mathrm{an}^d \simeq \mathrm{hocolim}_{K \in \mathcal{D}^\circ_\bullet} \mathcal{M}\mathrm{an}^d_K .$$

Note that for each $d \geq 0$, the construction $M \mapsto M \times [0, 1]$ determines a stabilization map $\mathcal{M}\mathrm{an}^d_K \to \mathcal{M}\mathrm{an}^{d+1}_K$, depending functorially on K. Set $\mathcal{M}\mathrm{an}^\infty_K = \varinjlim_{d \geq 0} \mathcal{M}\mathrm{an}^d_K$. It follows from the above analysis that the canonical map

$$\mathrm{hocolim}_{K \in \mathcal{D}^\circ_\bullet} \mathcal{M}\mathrm{an}^\infty_K \to \mathcal{M}\mathrm{an}^\infty$$

is a homotopy equivalence. Moreover, the composite map

$$\mathrm{hocolim}_{K \in \mathcal{D}^\circ_\bullet} \mathcal{M}\mathrm{an}^\infty_K \to \mathcal{M}\mathrm{an}^\infty \xrightarrow{\theta_\infty} \varinjlim_{d \geq 0} \mathcal{M}$$

is given by the composition

$$\mathrm{hocolim}_{K \in \mathcal{D}^\circ_\bullet} \mathcal{M}\mathrm{an}^\infty_K \to \mathrm{hocolim}_{K \in \mathcal{D}^\circ_\bullet} * \simeq \mathcal{M} \to \varinjlim_{d \geq 0} \mathcal{M},$$

where the last map is a homotopy equivalence (since the stabilization map $\mathcal{M} \xrightarrow{\times [0,1]} \mathcal{M}$ is a homotopy equivalence). Consequently, to prove Theorem 3, it will suffice to verify the following:

Proposition 8. *For every finite polyhedron K, the Kan complex $\mathcal{M}\mathrm{an}^\infty_K$ is contractible.*

Warning 9. It is natural to try to prove Proposition 8 by showing that the spaces $\mathcal{M}\mathrm{an}^d_K$ become highly connected as d becomes large. However, this is not necessarily true. For example, take K to be a single point, so that $\mathcal{M}\mathrm{an}^d_K$ is a classifying space for compact contractible framed PL manifolds. Note that any compact contractible PL d-manifold M admits a framing; if $\mathcal{M}\mathrm{an}^d_K$ were connected, then M would need to be PL homeomorphic to a disk of dimension d. However, this is not necessarily the case: if $d \geq 6$, then any closed PL manifold B with the homology of a $(d-1)$-sphere bounds a contractible PL manifold M of dimension d ([2]). If B is not simply connected, then M cannot be homeomorphic to a disk.

Following [3], we introduce an auxiliary condition to rule out the behavior of Warning 9.

Definition 10. Let K be a finite polyhedron. A *d-manifold thickening of K* is a cell-like PL map $\pi : M \to K$, where M is a framed PL manifold of dimension d, having the additional property that for each point $x \in K$, the intersection $\partial M \cap \pi^{-1}\{x\}$ is simply connected. We let $T^d(K)$ denote the simplicial subset of $\mathcal{M}\mathrm{an}^d_K$ whose k-simplices are triples (E, ρ, q) where $q : E \to K$ is a d-manifold thickening of K on each fiber.

Note that if $\pi : M \to K$ is any cell-like map and $\pi' : M \times [0, 1] \to K$ is the composition of π with the projection, then $\partial(M \times [0, 1]) \cap \pi'^{-1}\{x\}$ is homotopy equivalent to the suspension of $\partial(M) \cap \pi^{-1}\{x\}$ for each $x \in K$. It follows that the stabilization map $\mathcal{M}\mathrm{an}^d_K \to \mathcal{M}\mathrm{an}^{d+1}_K$ carries $T^d(K)$ into $T^{d+1}(K)$, and that the three-fold iterate of the stabilization map carries *all* of $\mathcal{M}\mathrm{an}^d_K$ into $T^{d+3}(K)$. We may therefore identify $\mathcal{M}\mathrm{an}^\infty_K$ with the direct limit $\varinjlim_{d \geq 0} T^d(K)$. The main step in the proof of Theorem 3 will be to verify the following:

Proposition 11. *Let K be a finite polyhedron and let n be an integer. Then for every sufficiently large integer d (where the meaning of "sufficiently large" depends on K and n), the space of d-manifold thickenings $T^d(K)$ is n-connected.*

References

[1] Haefliger, A. and V. Poenaru. *La classification des immersions combinatories.* Publ. Math. IHES, no. 23, pp. 75-91.

[2] Price, T. M. *Compact, contractible n-manifolds and their boundaries.* Michigan Math. J. Volume 18, Issue 4 (1971), 331-341.

[3] Waldhausen, F., B. Jahren and J. Rognes. *Spaces of PL Manifolds and Categories of Simple Maps.*