Overview of Part 3 (Lecture 34)

December 3, 2014

We begin with the following:

Question 1. Let X be a space. Under what circumstances does X have the homotopy type of a compact manifold M?

The answer to this question depends heavily on exactly what sorts of manifolds we allow. If we require M to be a *closed* manifold, then this places strong constraints on the space X: it must satisfy Poincare duality. Let us therefore be a bit more liberal, and allow M to be a manifold with boundary. In this case, there is a simple necessary condition:

Claim 2. *Let M be a compact manifold with boundary. Then M is homotopy equivalent to a finite CW complex.*

Proof. If M is a piecewise-linear manifold, this is clear: any finite polyhedron M is actually *homeomorphic* to a finite CW complex, with a CW structure given by a choice of triangulation of M.

If M is a smooth manifold, then we can choose a Whitehead triangulation of M and thereby reduce to the piecewise-linear case.

If M is merely a topological manifold, then Claim 2 is nontrivial (as we remarked in Lecture 2). It is easy to see that M is a *finitely dominated* space, but it is not easy to show that its Wall finiteness obstruction vanishes. This follows from a theorem of Chapman, which asserts that for any compact ANR M, there exists a finite polyhedron N and a homeomorphism $M \times Q \simeq N \times Q$, where Q is the Hilbert cube. □

This necessary condition turns out to be sufficient. First, we note that any CW complex is homotopy equivalent to a finite polyhedron (since the collection of homotopy types of finite polyhedra contains the empty and one point space and is closed under the formation of homotopy pushouts). If X is a finite polyhedron, then we can choose a PL embedding $X \hookrightarrow \mathbb{R}^n$ for n sufficiently large.

Definition 3. Let $X \subseteq \mathbb{R}^n$ be a finite polyhedron. A *regular neighborhood* of X is a finite polyhedron $N \subseteq \mathbb{R}^n$ satisfying the following conditions:

- The polyhedron X is contained in the interior of N.
- The polyhedron N is a PL manifold with boundary.
- The inclusion of X into N can be written as a composition of (polyhedral) elementary expansions (in other words, N "collapses" onto X).

Theorem 4. *Let $X \subseteq \mathbb{R}^n$ be a finite polyhedron. Then there exists a regular neighborhood of X in $\mathbb{R}^n$. Moreover, if N and N' are two regular neighborhoods of X in $\mathbb{R}^n$, then there exists a PL homeomorphism of N with N' which is the identity on X (in fact, one can be more precise: there is a PL isotopy of $\mathbb{R}^n$ which carries N to N', and is the identity on X).*

Theorem 4 supplies an answer to Question 1:

Corollary 5. *Let X be a finite CW complex. Then there exists a homotopy equivalence $X \simeq N$, where N is a compact manifold with boundary.*

For the existence part of Theorem 4, choose a cube $C \subseteq \mathbb{R}^n$ which contains X in its interior, and choose a triangulation $\Sigma(C)$ of C which restricts to a triangulation $\Sigma(X)$ of X. Let $\Sigma'(C)$ denote the barycentric subdivision of this triangulation and $\Sigma''(C)$ the barycentric subdivision of $\Sigma'(C)$. We can then take N to be the union of those (closed) simplices of $\Sigma''(C)$ which intersect X. For a proof that this construction works (and of the uniqueness asserted in Theorem 4), we refer the reader to [3].

Exercise 6. Let X be the boundary of a 2-simplex σ embedded in $\mathbb{R}^2$. Then we can choose a triangulation of $\mathbb{R}^2$ which includes σ as a simplex. Contemplate this example to appreciate the need to take a *second* barycentric subdivision in the construction sketched above.

The deduction of Corollary 5 from Theorem 4 actually yields more precise information:

- (a) The homotopy equivalence $X \simeq N$ can be chosen to be a *simple* homotopy equivalence (after replacing X by a finite polyhedron, we can arrange that the inclusion $X \hookrightarrow N$ is a composition of elementary expansions).
- (b) The manifold N can be chosen to be piecewise-linear.
- (c) The manifold N can be chosen to have trivial tangent microbundle T_N.

Remark 7. If N is a PL manifold with boundary, then the projection map $N \times N \to N$ is not a PL microbundle in the sense of the previous lecture, because N is not locally homeomorphic to $\mathbb{R}^n$ on its boundary. However, if N° denotes the interior of N, then the projection $N \times N^\circ \to N^\circ$ is a PL microbundle over N°, and the inclusion $N^\circ \hookrightarrow N$ is a homotopy equivalence; this determines a PL microbundle on N, which we denote by T_N and refer to as the *tangent microbundle* to N.

Definition 8. Let N be a PL manifold of dimension n (possibly with boundary). A *parallelization* of N is a microbundle equivalence of T_N with $\mathbb{R}^n \times N$.

More generally, suppose that $p : E \to B$ is a PL fiber bundle whose fibers are PL manifolds of dimension n. Then the projection map $E \times_B E \to E$ can be regarded as a PL microbundle over E (at least away from the boundary), which we denote by $T_{E/B}$. A *parallelization* of $E \to B$ is an equivalence of microbundles $T_{E/B} \simeq E \times R^n$.

Construction 9. For each integer $n \geq 0$, we define a simplicial set $\mathcal{M}^n$ as follows: a k-simplex of $\mathcal{M}^n$ consists of a finite polyhedron $E \subseteq \Delta^k \times \mathbb{R}^\infty$ for which the projection $E \to \Delta^k$ is a PL fiber bundle whose fibers are PL manifolds (with boundary) of dimension n, together with a parallelization of E.

In what follows, we will generally abuse notation and identify k-simplices fo $\mathcal{M}^n$ with the PL fiber bundle $E \to \Delta^k$, regarding the embedding $E \hookrightarrow \Delta^k \times \mathbb{R}^\infty$ and the parallelization of E as implicitly specified.

The construction

$$(E \to \Delta^k) \mapsto (E \times [0,1] \to \Delta^k)$$

determines a *stabilization map* $\sigma_n : \mathcal{M}^n \to \mathcal{M}^{n+1}$. We let $\mathcal{M}^\infty$ denote the direct limit of the sequence

$$\mathcal{M}^0 \to \mathcal{M}^1 \to \mathcal{M}^2 \to \cdots .$$

Note that every fiber bundle $E \to \Delta^k$ as in Construction 9 is, in particular, a fibration. Consequently, we have evident forgetful functors $\theta_n : \mathcal{M}^n \to \mathcal{M}$. Moreover, the diagrams

$$\begin{array}{ccc} \mathcal{M}^n & \xrightarrow{\sigma_n} & \mathcal{M}^{n+1} \\ & {\scriptstyle \theta_n} \searrow \quad \swarrow {\scriptstyle \theta_{n+1}} & \\ & \mathcal{M} & \end{array}$$

commute up to *canonical* homotopy, so that the maps θ_n can be amalgamated to a map

$$\theta : \mathcal{M}^{\infty} \to \mathcal{M}.$$

Our objective in the next part of this course is to prove the following result:

Theorem 10. *The map $\theta : \mathcal{M}^{\infty} \to \mathcal{M}$ is a homotopy equivalence.*

Remark 11. The simplest consequence of Theorem 10 is that the map θ is surjective on connected components. This asserts that every point of $\mathcal{M}$ (given by a finite polyhedron X) can be connected by a path in $\mathcal{M}$ (that is, a simple homotopy equivalence) to a point lying in the image of θ (that is, a polyhedron which is a parallelized PL manifold with boundary). This is equivalent to the contents of Corollary 5, together with the strengthenings (a), (b), and (c) indicated above.

Theorem 10 is a stronger result: it asserts that for a finite polyhedron X, not only can we choose a simple homotopy equivalence $X \simeq N$ to a parallelized PL manifold N, but that modulo "stabilization" (given by iterated product with $[0,1]$), the PL manifold N is unique up to a contractible space of choices. In particular, if N and N' are parallelized PL manifolds equipped with simple homotopy equivalences to X (and therefore to each other), then we can find integers $a, b \geq 0$ and a PL homeomorphism $N \times [0,1]^a \simeq N' \times [0,1]^b$. This recovers a form of the uniqueness asserted in Theorem 4.

We can summarize the situation informally by saying that Theorem 10 can be regarded as providing a *parametrized* version of regular neighborhood theory, at least after stabilization.

Combining Theorem 10 with the main result of the second part of this course, we obtain the following:

Corollary 12. *Let X be a finitely dominated space. Then there is a homotopy equivalence*

$$\mathcal{M}^{\infty} \times_{\mathcal{M}^h} \{X\} \simeq \Omega^{\infty}(X_+ \wedge A(*)) \times_{\Omega^{\infty} A(X)} \{[X]\}.$$

One can ask analogous questions in the setting of *smooth* manifolds. For each $n \geq 0$, one can introduce a simplicial set $\mathcal{M}^n_{\mathrm{sm}}$ analogous to $\mathcal{M}^n$, whose k-simplices are given by *smooth* submersions $E \to \Delta^k$ of parallelized manifolds. The direct limit $\mathcal{M}^{\infty}_{\mathrm{sm}} = \varinjlim \mathcal{M}^n_{\mathrm{sm}}$ maps to $\mathcal{M}$ via a map $\theta_{\mathrm{sm}} : \mathcal{M}^{\infty}_{\mathrm{sm}} \to \mathcal{M}$, but the map θ_{sm} is *not* a homotopy equivalence. However, we will show that the relationship between $\mathcal{M}^{\infty}_{\mathrm{sm}}$ and $\mathcal{M}^h$ is also governed by an A-theory assembly map. More precisely, we will prove the following version of Corollary 12:

Variant 13. Let X be a finitely dominated space. Then the homotopy fiber product $\mathcal{M}^{\infty}_{\mathrm{sm}} \times_{\mathcal{M}^h} \{X\}$ can be identified with the homotopy fiber of the map $u : \Omega^{\infty}\Sigma^{\infty}_{+} X \to \Omega^{\infty} A(X)$ over the point $[X] \in \Omega^{\infty} A(X)$. Here u is given by composing the A-theory assembly map with map $\Sigma^{\infty}_{+} X \to X_+ \wedge A(*)$ determined by the unit map $S \to A(*)$.

To understand the relationship between Corollary 12 and Variant 13, let us consider the problem of *smoothing* a PL manifold with boundary. For any PL n-manifold with boundary M, the tangent microbundles of M and ∂M are classified by a map of pairs

$$\chi : (M, \partial M) \to (\mathrm{BPL}(n), \mathrm{BPL}(n-1)).$$

In order to choose a smooth structure on M, we need to factor this classifying map through the pair $(\mathrm{BO}(n), \mathrm{BO}(n-1))$.

If we assume that M is equipped with a parallelization, then the situation simplifies: a parallelization of M is a nullhomotopy of the map $M \to \mathrm{BPL}(n)$, so that we can regard χ as a map from ∂M to the homotopy fiber $\mathrm{fib}(\mathrm{BPL}(n-1) \to \mathrm{BPL}(n)) = \mathrm{PL}(n)/\mathrm{PL}(n-1)$. In order to lift M to a (parallelized) smooth manifold, we need to factor this map through the quotient $\mathrm{fib}(\mathrm{BO}(n-1) \to \mathrm{BO}(n)) \simeq O(n)/O(n-1) \simeq S^{n-1}$.

Note that the sequence of spaces $\{O(n+1)/O(n)\}_{n \geq 0}$ can be regarded as a prespectrum which represents the sphere spectrum S. There is an analogous result for the groups $\mathrm{PL}(n)$:

Theorem 14. *The sequence of spaces* $\{\mathrm{PL}(n+1)/\mathrm{PL}(n)\}_{n\geq 0}$ *can be regarded as a prespectrum, whose associated spectrum is* $A(*)$.

For any parallelized PL n-manifold M, the classifying map χ above determines a map

$$\partial M \to \mathrm{PL}(n)/\mathrm{PL}(n-1) \to \Omega^{\infty-n+1}A(*),$$

whch we can regard as an element of $\Omega^{\infty-n+1}A(*)^{\partial M}$.

Theorem 15. *The boundary map*

$$\Omega^{\infty-n+1}A(*)^{\partial M} \to \Omega^{\infty-n}A(*)^{M/\partial M}$$

carries the classifying map χ *defined above to the image of* $\langle M\rangle$ *under the Atiyah duality map* $\Omega^{-n}A(*)^{M/\partial M} \simeq (M_+ \wedge A(*))$.

We will see that Variant 13 is a formal consequence of Theorem 15, since the natural maps $\mathrm{BO}(n) \to \mathrm{BPL}(n)$ give rise to a map of prespectra

$$S^n \simeq O(n+1)/O(n) \to \mathrm{PL}(n+1)/\mathrm{PL}(n)$$

which represents the unit map $S \to A(*)$.

References

[1] Chapman, T.A. *Piecewise Linear Fibrations.*

[2] Dywer, W., Weiss, M., and B. Williams. *A Parametrized Index Theorem for the Algebraic K-Theory Euler Class.*

[3] Hudson, J.F.P. *Piecewise-Linear Topology.*

[4] Waldhausen, F., B. Jahren and J. Rognes. *Spaces of PL Manifolds and Categories of Simple Maps.*