Digression: Microbundles (Lecture 33)

December 1, 2014

Let M be a smooth manifold. To each point $x \in M$, one can associate a real vector space $T_{M,x}$, called the *tangent space* to x in M. The union $\bigcup_{x \in M} T_{M,x}$ can be regarded as a vector bundle over M, which we denote by T_M and refer to as the *tangent bundle* of M.

If M is a topological manifold which does not have a smooth structure, then one generally cannot associate to M a tangent vector bundle. To remedy the situation, Milnor introduced the theory of *microbundles*.

Definition 1 (Milnor)**.** Let B be a topological space. An *topological microbundle* on X (of rank n) is a map $p : E \to B$ equipped with a section $s : B \to E$ satisfying the following condition:

$(*)$ For every point $x \in B$, there exists a neighborhood of $U \subseteq B$ containing x and an open subset of E homeomorphic to $U \times \mathbb{R}^n$, such that the section s can be identified with the zero section $U \simeq U \times \{0\} \hookrightarrow U \times \mathbb{R}^n$.

An *equivalence* of microbundles E and E' over B is a homeomorphism $h : U \simeq U'$ fitting into a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & U' \\ & \searrow \quad \swarrow & \\ & X, & \end{array}$$

where U is an open subset of E containing the image of the section $s : B \to E$, U' is an open subset of E' containing the image of $s' : B \to E'$, and we have $h \circ s = s'$.

Example 2. Let M be a topological manifold. The *tangent microbundle* T_M is defined to be the product $M \times M$, mapping to M via the projection $\pi_1 : M \times M \to M$, with section $s : M \to M \times M$ given by the diagonal map.

Variant 3. In Definition 1, we can require E and B to be (not necessarily finite) polyhedra and all of the relevant homeomorphisms to be PL maps. This gives us the notion of a *PL microbundle* over a polyhedron B.

If M is a PL manifold, then the product $M \times M$ can be regarded as a PL microbundle over M (denoted also by T_M).

Variant 4. In Definition 1, we can require p to be a smooth submersion: that is, we can require E to admit a system of charts homeomorphic to $U \times \mathbb{R}^n$ where U is an open subset of B, where the transition functions between charts are continuous in the first variable and infinitely differentiable in the second. This leads to the notion of a *smooth microbundle* over B.

If M is a smooth manifold, then the product $M \times M$ can be regarded as a smooth microbundle over M (again denoted by T_M).

Example 5. Let $\mathcal{E}$ be a vector bundle over a space B. Then $\mathcal{E}$ can be regarded as a smooth microbundle over B (since linear maps are infinitely differentiable).

Conversely, if $E \to B$ is a smooth microbundle with zero section $s : B \to E$, then the relative tangent bundle $T_{E/B}$ is a vector bundle over E whose pullback $s^* T_{E/B}$ can be regarded as a vector bundle over B. This construction determines a map

$$\{ \text{ smooth microbundles over } B\}/ \text{ equivalence } \to \{ \text{ vector bundles over } B\}/ \text{ isomorphism } .$$

It is easy to see that this construction is left inverse to the construction which regards each vector bundle as a smooth microbundle. If B is paracompact, then it is also a right inverse: in other words, any smooth microbundle $E \to B$ is equivalent (as a smooth microbundle) to the vector bundle $s^* T_{E/B}$. One can produce an equivalence by choosing a Riemannian metric on each fiber of E (depending continuously on B) and using it to define a map $s^* T_{E/B} \to E$ by means of the "exponential spray".

Definition 6. Let B be a topological space. A $\mathrm{Top}(n)$-*bundle over* B is a fiber bundle $p : E \to B$ whose fibers are homeomorphic to $\mathbb{R}^n$.

Remark 7. If B is paracompact (which we should always assume here), then any $\mathrm{Top}(n)$-bundle $E \to B$ admits a section and can therefore be regarded as a topological microbundle over B.

Variant 8. If B is a polyhedron, then we also have the notion of a $\mathrm{PL}(n)$-bundle over B: that is, a map of polyhedra $E \to B$ which is locally PL isomorphic to the product of the base with $\mathbb{R}^n$. Any $\mathrm{PL}(n)$-bundle can be regarded as a PL microbundle.

Theorem 9 (Kister-Mazur)**.** *Let B be paracompact. Then the natural map*

$$\{ \mathrm{Top}(n)\text{-bundles over } B\}/ \text{ isomorphism } \to \{ \text{ rank } n \text{ microbundles over } B\}/ \text{ equivalence}$$

is bijective. In other words, every topological microbundle admits an essentially unique refinement to a $\mathrm{Top}(n)$*-bundle.*

Variant 10 (Kuiper-Lashof)**.** Let B be a polyhedron. Then the natural map

$$\{ \mathrm{PL}(n)\text{-bundles over } B \}/ \text{ isomorphism } \to \{ \text{ rank } n \text{ PL microbundles over } B \}/ \text{ equivalence}$$

is bijective. In other words, every PL microbundle admits an essentially unique refinement to a $\mathrm{PL}(n)$-bundle.

Example 11. Let M be a (paracompact) topological manifold. Then every point $x \in M$ admits an open neighborhood U_x which is homeomorphic to a Euclidean space $\mathbb{R}^n$. It follows from Theorem 9 that the open sets U_x can be chosen "uniformly" so that the disjoint union $\amalg_{x \in M} U_x = \{(x, y) \in M \times M : y \in U_x\}$ is an open subset of $M \times M$ (to get a feeling for the content of Theorem 9, try proving this directly).

We can regard $\mathrm{Top}(n)$ and $\mathrm{PL}(n)$ as simplicial groups, whose k-simplices are homeomorphisms (required to be PL in the second case) of $\mathbb{R}^n \times \Delta^k$ with itself which commute with projection onto the second factor. The classifying spaces $\mathrm{BTop}(n)$ and $\mathrm{BPL}(n)$ classify $\mathrm{Top}(n)$-bundles and $\mathrm{PL}(n)$-bundles, respectively. By virtue of the above results, we can think of $\mathrm{BTop}(n)$ and $\mathrm{BPL}(n)$ as classifying spaces for topological and PL microbundles of rank n, respectively. Similarly, $\mathrm{BO}(n)$ is a classifying space for smooth microbundles of rank n.