Digression: Microbundles (Lecture 33)

December 1, 2014

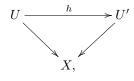
Let M be a smooth manifold. To each point $x \in M$, one can associate a real vector space $T_{M,x}$, called the *tangent space* to x in M. The union $\bigcup_{x \in M} T_{M,x}$ can be regarded as a vector bundle over M, which we denote by T_M and refer to as the *tangent bundle* of M.

If M is a topological manifold which does not have a smooth structure, then one generally cannot associate to M a tangent vector bundle. To remedy the situation, Milnor introduced the theory of *microbundles*.

Definition 1 (Milnor). Let B be a topological space. An topological microbundle on X (of rank n) is a map $p: E \to B$ equipped with a section $s: B \to E$ satisfying the following condition:

(*) For every point $x \in B$, there exists a neighborhood of $U \subseteq B$ containing x and an open subset of E homeomorphic to $U \times \mathbb{R}^n$, such that the section s can be identified with the zero section $U \simeq U \times \{0\} \hookrightarrow U \times \mathbb{R}^n$.

An equivalence of microbundles E and E' over B is a homeomorphism $h: U \simeq U'$ fitting into a commutative diagram



where U is an open subset of E containing the image of the section $s: B \to E, U'$ is an open subset of E' containing the image of $s': B \to E'$, and we have $h \circ s = s'$.

Example 2. Let M be a topological manifold. The *tangent microbundle* T_M is defined to be the product $M \times M$, mapping to M via the projection $\pi_1 : M \times M \to M$, with section $s : M \to M \times M$ given by the diagonal map.

Variant 3. In Definition 1, we can require E and B to be (not necessarily finite) polyhedra and all of the relevant homeomorphisms to be PL maps. This gives us the notion of a *PL microbundle* over a polyhedron B.

If M is a PL manifold, then the product $M \times M$ can be regarded as a PL microbundle over M (denoted also by T_M).

Variant 4. In Definition 1, we can require p to be a smooth submersion: that is, we can require E to admit a system of charts homeomorphic to $U \times \mathbb{R}^n$ where U is an open subset of B, where the transition functions between charts are continuous in the first variable and infinitely differentiable in the second. This leads to the notion of a *smooth microbundle* over B.

If M is a smooth manifold, then the product $M \times M$ can be regarded as a smooth microbundle over M (again denoted by T_M).

Example 5. Let \mathcal{E} be a vector bundle over a space B. Then \mathcal{E} can be regarded as a smooth microbundle over B (since linear maps are infinitely differentiable).

Conversely, if $E \to B$ is a smooth microbundle with zero section $s : B \to E$, then the relative tangent bundle $T_{E/B}$ is a vector bundle over E whose pullback $s^*T_{E/B}$ can be regarded as a vector bundle over B. This construction determines a map

{ smooth microbundles over B}/ equivalence \rightarrow { vector bundles over B}/ isomorphism.

It is easy to see that this construction is left inverse to the construction which regards each vector bundle as a smooth microbundle. If B is paracompact, then it is also a right inverse: in other words, any smooth microbundle $E \to B$ is equivalent (as a smooth microbundle) to the vector bundle $s^*T_{E/B}$. One can produce an equivalence by choosing a Riemannian metric on each fiber of E (depending continuously on B) and using it to define a map $s^*T_{E/B} \to E$ by means of the "exponential spray".

Definition 6. Let B be a topological space. A Top(n)-bundle over B is a fiber bundle $p: E \to B$ whose fibers are homeomorphic to \mathbb{R}^n .

Remark 7. If B is paracompact (which we should always assume here), then any Top(n)-bundle $E \to B$ admits a section and can therefore be regarded as a topological microbundle over B.

Variant 8. If B is a polyhedron, then we also have the notion of a PL(n)-bundle over B: that is, a map of polyhedra $E \to B$ which is locally PL isomorphic to the product of the base with \mathbb{R}^n . Any PL(n)-bundle can be regarded as a PL microbundle.

Theorem 9 (Kister-Mazur). Let B be paracompact. Then the natural map

 $\{ \operatorname{Top}(n) \text{-bundles over } B \} / \text{ isomorphism } \rightarrow \{ \text{ rank } n \text{ microbundles over } B \} / \text{ equivalence} \}$

is bijective. In other words, every topological microbundle admits an essentially unique refinement to a Top(n)-bundle.

Variant 10 (Kuiper-Lashof). Let B be a polyhedron. Then the natural map

 $\{ PL(n) \text{-bundles over } B \} / \text{ isomorphism } \rightarrow \{ \text{ rank } n \text{ PL microbundles over } B \} / \text{ equivalence}$

is bijective. In other words, every PL microbundle admits an essentially unique refinement to a PL(n)-bundle.

Example 11. Let M be a (paracompact) topological manifold. Then every point $x \in M$ admits an open neighborhood U_x which is homeomorphic to a Euclidean space \mathbb{R}^n . It follows from Theorem 9 that the open sets U_x can be chosen "uniformly" so that the disjoint union $\coprod_{x \in M} U_x = \{(x, y) \in M \times M : y \in U_x\}$ is an open subset of $M \times M$ (to get a feeling for the content of Theorem 9, try proving this directly).

We can regard Top(n) and PL(n) as simplicial groups, whose k-simplices are homeomorphisms (required to be PL in the second case) of $\mathbb{R}^n \times \Delta^k$ with itself which commute with projection onto the second factor. The classifying spaces BTop(n) and BPL(n) classify Top(n)-bundles and PL(n)-bundles, respectively. By virtue of the above results, we can think of BTop(n) and BPL(n) as classifying spaces for topological and PL microbundles of rank n, respectively. Similarly, BO(n) is a classifying space for smooth microbundles of rank n.