Higher Torsion (Lecture 27)

November 5, 2014

Let Poly denote the ordinary category of finite polyhedra, and let $\mathcal{S}$ denote the ∞-category of spaces. Over the last few lectures, we have studied the functor

$$K_\Delta : \mathrm{Poly} \to \mathcal{S}.$$

given by

$$K_\Delta(X) = |K(\mathrm{Shv}_{PL}^{\Delta^\bullet}(X \times \Delta^\bullet))|.$$

Since every finite polyhedron has an underlying topological space, there is a forgetful functor $\iota : \mathrm{Poly} \to \mathcal{S}$. Let us (temporarily) use the notation $\iota_! K_\Delta$ to denote the left Kan extension of K_Δ along ι. This left Kan extension can be computed in two steps:

- First, we can form the left Kan extension of ι along the forgetful functor $\mathrm{Poly} \to \mathcal{S}^{\mathrm{fin}}$, where $\mathcal{S}^{\mathrm{fin}}$ is the ∞-category of finite spaces. Since K_Δ is homotopy invariant, this is equivalent to lifting K_Δ along the fully faithful embedding
$$\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}, \mathcal{S}) \to \mathrm{Fun}(\mathrm{Poly}, \mathcal{S}).$$
- We then form the left Kan extension along the fully faithful embedding $\mathcal{S}^{\mathrm{fin}} \to \mathcal{S}$. This is the process of formally extending a functor $\mathcal{S}^{\mathrm{fin}} \to \mathcal{S}$ to a functor $\mathcal{S} \to \mathcal{S}$ so that it commutes with filtered colimits.

It follows from this analysis that the restriction of $i_! K_\Delta$ to Poly agrees with the original functor K_Δ. We will henceforth abuse notation by denoting the functor $i_! K_\Delta$ also by K_Δ, so that we view K_Δ as a functor from spaces to spaces. The main theorem of the previous lectures gives us an explicit description of this functor: it is the domain of the assembly map in Waldhausen A-theory. That is, we have

$$K_\Delta(X) \simeq \Omega^\infty(X_+ \wedge A(*)).$$

We can use this identification to produce some $A(*)$-homology classes. Let X be a space, and suppose we are given a finite polyhedron Y, a map $f : Y \to X$, and a constructible sheaf $\mathcal{F}$ on Y (with values in the ∞-category of finite spectra). Then $\mathcal{F}$ is an object of $\mathrm{Shv}_{PL}(Y)$ and therefore determines a point of $K(\mathrm{Shv}_{PL}(Y))$, and therefore also of $K_\Delta(Y)$. Using the map f, we obtain a point of $K_\Delta(X)$ which we will denote by $\langle Y, \mathcal{F} \rangle$. In the special case where $\mathcal{F}$ is the constant sheaf on Y (with value the sphere spectrum), we will denote this point simply by $\langle Y \rangle$.

We have an assembly map $K_\Delta(X) \to \Omega^\infty A(X)$. Unwinding the definitions, we see that this assembly map carries $\langle Y, \mathcal{F} \rangle$ to $[\mathcal{F}']$, where $\mathcal{F}'$ is the local system of spectra on X which corepresents the functor

$$\mathrm{Sp}^X \to \mathrm{Sp}$$

$$\mathcal{G} \mapsto \Gamma(Y, \mathcal{F} \wedge f^* \mathcal{G})$$

(here Γ denotes the global sections functor). In the special case where $\mathcal{F}$ is the constant sheaf, this functor is given by

$$\Gamma(Y, f^* \mathcal{G}) = \mathrm{Map}_{\mathrm{Sp}^Y}(\underline{S}_Y, f^* \mathcal{G}) = \mathrm{Map}_{\mathrm{Sp}^X}(f_! \underline{S}_Y, f^* \mathcal{G}).$$

It follows that $\mathcal{F}' \simeq f_! \underline{S}_Y$ (where $f_!$ denotes the left adjoint to pullback on local systems), so that $[\mathcal{F}']$ can be identified with the point $[Y] \in \Omega^\infty A(X)$ studied in Lecture 21. This analysis proves the following:

Proposition 1. *Let X be any space. For any finite polyhedron Y and any map $f : Y \to X$, the assembly map $K_\Delta(X) \to \Omega^\infty A(X)$ carries $\langle Y \rangle \in K_\Delta(X)$ to $[Y] \in \Omega^\infty A(X)$.*

All of the preceding considerations can be generalized to "allow parameters". Let us be more precise. Fix a topological space X. We define Kan complexes $\mathcal{M}_X$ and $\mathcal{M}^h_X$ as follows:

- The n-simplices of $\mathcal{M}_X$ are finite polyhedra $Y \subseteq \Delta^n \times \mathbb{R}^\infty$ equipped with a map $f : Y \to X$, for which the projection $Y \to \Delta^n$ is a PL fibration.
- The n-simplices of $\mathcal{M}^h_X$ are subspaces $Y \subseteq \Delta^n \times \mathbb{R}^\infty$ equipped with a map $f : Y \to X$ for which the projection $Y \to \Delta^n$ is a fibration with finitely dominated fibers.

The construction $(Y \to X) \mapsto [Y]$ can be naturally refined to a map of Kan complexes $\mathcal{M}^h_X \to \Omega^\infty A(X)$, and the construction $(Y \to X) \mapsto \langle Y \rangle$ can be naturally refined to a map of Kan complexes $\mathcal{M}_X \to K_\Delta(X)$. Repeating the analysis that preceded Proposition 1, we obtain the following refinement:

Proposition 2. *Let X be any space. Then the diagram*

$$\begin{array}{ccc} \mathcal{M}_X & \longrightarrow & K_\Delta(X) \\ \downarrow & & \downarrow \\ \mathcal{M}^h_X & \longrightarrow & \Omega^\infty A(X) \end{array}$$

commutes (up to canonical homotopy).

Let us now suppose that the space X itself is finitely dominated. In this case, the Kan complex $\mathcal{M}^h_X$ contains a contractible path component whose vertices are *homotopy equivalences* $Y \to X$. Let us denote this path component by $\mathcal{M}^{h\circ}_X$. We have a diagram of homotopy pullback squares

$$\begin{array}{ccccc} \mathcal{M}_X \times_{\mathcal{M}^h_X} \mathcal{M}^{h\circ}_X & \longrightarrow & \mathcal{M}_X & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{M}^{h\circ}_X & \longrightarrow & \mathcal{M}^h_X & \longrightarrow & \mathcal{M}^h . \end{array}$$

In other words, the homotopy fiber of the map $\mathcal{M} \to \mathcal{M}^h$ over X can be identified with $\mathcal{M}_X \times_{\mathcal{M}^h_X} \mathcal{M}^{h\circ}_X$. Applying Proposition 2, we obtain a map

$$\begin{aligned} \mathcal{M} \times_{\mathcal{M}^h} \{X\} & \simeq \mathcal{M}_X \times_{\mathcal{M}^h_X} \mathcal{M}^{h\circ}_X \\ & \to K_\Delta(X) \times_{\Omega^\infty A(X)} \mathcal{M}^{h\circ}_X \\ & \to K_\Delta(X) \times_{\Omega^\infty A(X)} \{[X]\}. \end{aligned}$$

We can now give a more precise formulation of the main result of the second part of this course:

Theorem 3. *Let X be a finitely dominated space. Then the map*

$$\mathcal{M} \times_{\mathcal{M}^h} \{X\} \to K_\Delta(X) \times_{\Omega^\infty A(X)} \{[X]\}$$

is a homotopy equivalence.

Example 4. Theorem 3 implies that the homotopy fiber $\mathcal{M} \times_{\mathcal{M}^h} \{X\}$ is either empty (in case X has non-vanishing Wall obstruction) or a torsor for the infinite loop space

$$\operatorname{fib}(K_\Delta(X) \to \Omega^\infty A(X)) \simeq \Omega^{\infty+1} \operatorname{Wh}(X),$$

where $\mathrm{Wh}(X)$ denotes the (piecewise-linear) Whitehead spectrum of X.

If X itself is given as a finite polyhedron, then the space $\mathcal{M} \times_{\mathcal{M}^h} \{X\}$ has a canonical base point. In this case, we obtain a *canonical* homotopy equivalence

$$\tau : \mathcal{M} \times_{\mathcal{M}^h} \{X\} \simeq \Omega^{\infty+1} \mathrm{Wh}(X).$$

Note that the points of $\mathcal{M} \times_{\mathcal{M}^h} \{X\}$ can be identified with pairs (Y, f), where Y is a finite polyhedron and $f : Y \to X$ is a homotopy equivalence. If X itself is a finite polyhedron, then the "identity component" of $\mathcal{M} \times_{\mathcal{M}^h} \{X\}$ consists of those pairs (Y, f) where f is a *simple* homotopy equivalence. It follows from Theorem 3 that f is a simple homotopy equivalence if and only if a certain element $\tau(Y, f) \in \pi_1 \mathrm{Wh}(X)$ vanishes. If X is connected with fundamental group G, we have seen that there is a canonical isomorphism of $\pi_1 \mathrm{Wh}(X)$ with the Whitehead group $\mathrm{Wh}(G)$ of G, so we can identify $\tau(Y, f)$ with an element of $\mathrm{Wh}(G)$.

Proposition 5. *In the situation above, the element $\tau(Y, f) \in \mathrm{Wh}(G)$ coincides with the Whitehead torsion of the homotopy equivalence f (as defined in Lectures 3 and 4).*

Combining Proposition 5 with Theorem 3, we obtain another proof of the main result from Lecture 4: the homotopy equivalence $f : Y \to X$ is simple if and only if its Whitehead torsion vanishes. In other words, Proposition 5 allows us to regard Theorem 3 as a *generalization* of the main result of Lecture 4 (and, as we have already noted, Theorem 3 also generalizes the theory of the Wall obstruction).

Let us informally sketch a proof of Proposition 5. Without loss of generality, we may assume that Y and X have been equipped with triangulations that are compatible with the map f. Assume that X is connected with fundamental group G. We have a pair of points

$$\langle X \rangle, \langle Y \rangle \in K_\Delta(X),$$

having images $[X], [Y] \in \Omega^\infty A(X)$. Our assumption that f is a homotopy equivalence supplies an equivalence of local systems $f_! \underline{S}_Y \simeq \underline{S}_X$, which gives a path p joining $[X]$ and $[Y]$ in $\Omega^\infty A(X)$. This path gives a lift of $\langle X \rangle - \langle Y \rangle$ to the homotopy fiber

$$\Omega^{\infty+1} \mathrm{Wh}(X) \simeq \mathrm{fib}(K_\Delta(X) \to \Omega^\infty A(X),$$

and $\tau(Y, f)$ is the path component of this lift. Note that the map $\pi_0 K_\Delta(X) \to \pi_0 A(X)$ is injective, so that $\langle X \rangle$ and $\langle Y \rangle$ belong to the same path component of $\Omega^\infty A(X)$. If we choose a path q from $\langle X \rangle$ to $\langle Y \rangle$, then we can combine p with the image of q to form a closed loop in the space $\Omega^\infty A(X)$. This loop determines an element $\eta \in \pi_1 A(X) \simeq K_1(\mathbf{Z}[G])$, which is preimage of $\tau(Y, f)$ under the connecting homomorphism

$$\pi_1 A(X) \to \pi_0(\mathrm{fib}\, K_\Delta(X) \to \Omega^\infty A(X)).$$

Note that the element η depends on the choice of path q.

Let $\Sigma(X)$ and $\Sigma(Y)$ denote the set of simplices of X and Y, respectively. Let $\underline{S}_X$ and $\underline{S}_Y$ denote the constant sheaves (with value the sphere spectrum) on X and Y, respectively. For each simplex σ of X (or Y), let $\underline{S}_\sigma$ denote the constructible sheaf on X (or Y) taking the value S on σ and 0 elsewhere (in other words, the sheaf which is "extended by zero" from the interior of σ) and let $\underline{S}^n_\sigma$ denote the nth suspension of $\underline{S}_\sigma$. Note that

$$f_* \underline{S}_\sigma \simeq \underline{S}^{\dim f(\sigma) - \dim(\sigma)}_{f(\sigma)}.$$

For each $\sigma \in \Sigma(X) \cup \Sigma(Y)$, consider the point $e_\sigma \in K_\Delta(X)$ given by

$$e_\sigma = \begin{cases} [\underline{S}_\sigma] & \text{if } \Sigma \in \Sigma(X) \\ -[\underline{S}^{\dim f(\sigma) - \dim(\sigma)}_{f(\sigma)}] & \text{if } \Sigma \in \Sigma(Y) \end{cases}$$

Using the additivity theorem, we can choose a path from the difference $\langle X \rangle - \langle Y \rangle$ to the sum $\sum_{\sigma \in \Sigma(X) \cup \Sigma(Y)} e_\sigma$. Let E denote the union of the set of even-dimensional simplices of X and odd-dimensional simplices of Y, and let E' denote the union of the set of odd-dimensional simplices of X and even-dimensional simplices of Y. Note that $\pi_0 K_\Delta(X) \simeq \mathbf{Z}$, and e_σ belongs to the path component 1 if $\sigma \in E$ and the path component -1 if $\sigma \in E'$. Since f is a homotopy equivalence, X and Y have the same Euler characteristic and therefore E and E' have the same size. We may therefore choose a bijection $\beta : E \simeq E'$. For each $\sigma \in E$, we can choose a path q_σ in $K_\Delta(X)$ from $e_\sigma + e_{\beta(\sigma)}$ to the base point; note that these paths are ambiguous up to an element of $\pi_1 K_\Delta(X) \simeq G \oplus \mathbf{Z}/2\mathbf{Z}$. The sum of these paths determines a path q from $\langle X \rangle - \langle Y \rangle$ to the base point.

Unwinding the definitions, we see that the image of $[X] - [Y]$ in $K(\mathbf{Z}[G])$ can be represented by the relative cellular chain complex $C_*(X, Y; \mathbf{Z}[G])$. The given triangulations of X and Y determine a basis for $C_*(X, Y; \mathbf{Z}[G])$ as a $\mathbf{Z}[G]$-module, where the basis elements are ambiguous up to $\pm G$. We have two paths from $[C_*(X, Y; \mathbf{Z}[G])]$ to the base point of $K(\mathbf{Z}[G])$, given as follows:

- (a) The image of p determines a path from $[C_*(X, Y; \mathbf{Z}[G])]$ to the base point of $K(\mathbf{Z}[G])$ which arises from the observation that $C_*(X, Y; \mathbf{Z}[G])$ is an acyclic complex (because f is a homotopy equivalence), and therefore represents a zero object of the ∞-category $\mathrm{Rep}_{\mathbf{Z}[G]}$.

- (b) The image of the path q determines a path from $[C_*(X, Y; \mathbf{Z}[G])]$ to the base point of $K(\mathbf{Z}[G])$. After possibly modifying our choice of basis, we can arrange that this path is obtained by first invoking the additivity theorem to construct a path from $[C_*(X, Y; \mathbf{Z}[G])]$ to the point represented by the sum

$$\bigoplus_{\sigma \in \Sigma(X)} [\Sigma^{\dim(\sigma)} \mathbf{Z}[G]] + \bigoplus_{\sigma \in \Sigma(Y)} [\Sigma^{\dim(\sigma)+1} \mathbf{Z}[G]],$$

  and then connecting this latter sum to the base point by matching factors using the bijection β.

We are therefore reduced to the following statement, which we leave as a (tedious) exercise:

Exercise 6. Let R be a ring and let F_* be a bounded acyclic chain complex of free R-modules, where $\chi(F_*) = 0$ (the latter condition is automatic if R has the form $\mathbf{Z}[G]$). Suppose we have chosen a basis $\{e_i, e'_i\}$ for F_*, where each e_i is homogeneous of even degree d_i, and each e'_i is homogeneous of odd degree d'_i. Then the torsion $\tau(F_*) \in K_1(R) \simeq \pi_1 K(R)$ (as defined in Lecture 3) can be represented as the "difference" between two paths from $[F_*]$ to the base point of the space $K(R)$:

- (a) The path obtained from the observation that the chain complex F_* represents the zero object of Mod_R (since F_* is acyclic).

- (b) The path obtained by first using the additivity theorem to construct a path from $[F_*]$ to the sum $\sum_i [\Sigma^{d_i} R] \oplus [\Sigma^{d'_i} R]$, then connecting each $[\Sigma^{d_i} R] + [\Sigma^{d'_i} R]$ to the base point using the fact that d_i and d'_i have different parities.

References

[1] Waldhausen, F. *The Algebraic K-Theory of Spaces.*

[2] Dywer, W., Weiss, M., and B. Williams. *A Parametrized Index Theorem for the Algebraic K-Theory Euler Class.*