The Assembly Map, Part III (Lecture 26)

November 4, 2014

Our goal in this lecture is to complete the proof of the following result:

Proposition 1. *Suppose we are given a pushout diagram*

$$\begin{array}{ccc} X_{01} & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X \end{array}$$

in the ∞-category of finite spaces. Then the diagram

$$\begin{array}{ccc} K_\Delta(X_{01}) & \longrightarrow & K_\Delta(X_0) \\ \downarrow & & \downarrow \\ K_\Delta(X_1) & \longrightarrow & K_\Delta(X) \end{array}$$

is also a pushout square of E_∞-spaces.

To prove Proposition 1, it will be convenient to consider instead the diagram of connected deloopings

$$\begin{array}{ccc} \Omega^{-1}K_\Delta(X_{01}) & \longrightarrow & \Omega^{-1}K_\Delta(X_0) \\ \downarrow & & \downarrow \\ \Omega^{-1}K_\Delta(X_1) & \longrightarrow & \Omega^{-1}K_\Delta(X). \end{array}$$

We wish to compare $\Omega^{-1}K_\Delta(X)$ with the homotopy pushout of the rest of the diagram, which is computed as the geometric realization of a simplicial space

$$\mathrm{Bar}_\bullet(\Omega^{-1}K_\Delta(X_0), \Omega^{-1}K_\Delta(X_{01}), \Omega^{-1}K_\Delta(X_1))$$

given by

$$\mathrm{Bar}_n(\Omega^{-1}K_\Delta(X_0), \Omega^{-1}K_\Delta(X_{01}), \Omega^{-1}K_\Delta(X_1)) = (\Omega^{-1}K_\Delta(X_0)) \times (\Omega^{-1}K_\Delta(X_{01})^n) \times (\Omega^{-1}K_\Delta(X_1)).$$

As in the previous lecture, let us assume that X_0 and X_1 are finite polyhedra, that X_{01} is a subpolyhedron of each, and that X is given by the two-sided mapping cylinder

$$X_0 \amalg_{\{0\}\times X_{01}} ([0,1] \times X_{01}) \amalg_{\{1\}\times X_{01}} X_1.$$

For each $t \in (0,1)$, let X_t denote the subpolyhedron $\{t\} \times X_{01} \subseteq X$. Recall that a sheaf $\mathcal{F} \in \mathrm{Shv}^S_{PL}(X \times S)$ is *transverse to t* if $\mathcal{F}|_{X_t \times S}$ is ULA over S. More generally, if T is a finite subset of the open interval $(0,1)$, we will say that $\mathcal{F} \in \mathrm{Shv}^S_{PL}(X \times S)$ is *transverse* to T if it is transverse to t for each $t \in T$. Let $\mathrm{Shv}^{S,T}_{PL}(X \times S)$ denote the full subcategory of $\mathrm{Shv}^S_{PL}(X \times S)$ spanned by those sheaves which are transverse to T, and let $K^T_\Delta(X)$ denote the geometric realization of the simplicial space given by

$$K(\mathrm{Shv}^{\Delta^\bullet,T}_{PL}(X \times \Delta^\bullet).$$

We proved in the last lecture that if T has a single element, then $K^T_\Delta(X)$ can be identified with the product

$$K_\Delta(X_0) \times K_\Delta(X_{01}) \times K_\Delta(X_1).$$

One can carry out a similar argument for larger finite sets $T = \{t_1, \ldots, t_n\}$: every object of $\mathrm{Shv}^{S,T}_{PL}(X \times S)$ has a canonical filtration whose subquotients are sheaves which are supported either on $X_{t_i} \times S$ for $1 \leq i \leq n$, on $X_{<t_1} \times S$ (which contains $X_0 \times S$ as a deformation retract), on $X_{>t_n} \times S$ (which contains $X_1 \times S$ as a deformation retract), or on $X_{01} \times (t_i, t_{i+1}) \times S$ for $1 \leq i < n$. This analysis supplies a homotopy equivalence

$$K^T_\Delta(X) \simeq K_\Delta(X_0) \times K_\Delta(X_{01})^{2n-1} \times K_\Delta(X_1).$$

We can be more precise: the construction $T \mapsto K^T_\Delta(X)$ is contravariantly functorial in T, and there is a functorial identification

$$\Omega^{-1} K^T_\Delta(X) \simeq \mathrm{Bar}_{\alpha(T)}(\Omega^{-1} K_\Delta(X_0), \Omega^{-1} K_\Delta(X_{01}), \Omega^{-1} K_\Delta(X_1))$$

where $\alpha(T)$ is the finite linearly ordered set given by $T \times \{0,1\}$, equipped with the lexicographical ordering.

Lemma 2. *Let P be the poset of nonempty finite subsets of $(0,1)$ and let Δ be the category of finite nonempty linearly ordered sets. Then the construction*

$$T \mapsto T \times \{0,1\}$$

determines a right cofinal functor

$$\alpha : P \to \Delta.$$

Proof. Fix an object $[n] = \{0 < 1 < \cdots < n\} \in \Delta$; we wish to show that the category (in fact, poset) $Q = P \times_\Delta \Delta_{/[n]}$ is weakly contractible. For each $\epsilon > 0$, let $P_\epsilon \subseteq P$ denote the collection of all nonempty finite subsets of $(\epsilon, 1)$, and let Q_ϵ denote the inverse image in Q of P_ϵ. Then Q can be written as a directed union of the posets Q_ϵ. Consequently, to show that Q is weakly contractible, it will suffice to show that each of the inclusion maps $Q_\epsilon \hookrightarrow Q$ is nullhomotopic. We can identify the elements of Q with pairs (T, f) where $T \subseteq (0,1)$ is a nonempty finite set and $f : T \times \{0,1\} \to [n]$ is a monotone map. If $(T, f) \in Q_\epsilon$, then we have natural maps

$$(T, f) \hookrightarrow (T \cup \{\epsilon\}, f_+) \hookleftarrow (\{\epsilon\}, g),$$

where $g : \{\epsilon\} \times \{0,1\} \to [n]$ is the constant map taking the value 0 and f_+ is the amalgamation of the maps f and g. These maps determine a homotopy from the inclusion $Q_\epsilon \hookrightarrow Q$ to the constant map $Q_\epsilon \to \{(\{\epsilon\}, g)\} \subseteq Q$. □

It follows from Lemma 2 that the pushout

$$\Omega^{-1} K_\Delta(X_0) \amalg_{\Omega^{-1} K_\Delta(X_{01})} \Omega^{-1} K_\Delta(X_1)$$

(formed in the ∞-category of E_∞-spaces) can also be written as

$$\varinjlim_{T \subseteq (0,1)} \Omega^{-1} K^T_\Delta(X)$$

(formed in the ∞-category of spaces). We may therefore rewrite Proposition 1 as follows:

Proposition 3. *The canonical map*

$$\varinjlim_{T \subseteq (0,1)} \Omega^{-1} K^T_\Delta(X) \to \Omega^{-1} K_\Delta(X)$$

is a homotopy equivalence of spaces.

Note that the map of Proposition 3 can be obtained as the geometric realization of a map of bisimplicial spaces which is given (in bidegree (m, n)) by

$$\theta_m : \varinjlim_{T \subseteq (0,1)} S_n \operatorname{Shv}^{\Delta^m, T}_{PL}(X \times \Delta^m) \to S_n \operatorname{Shv}^{\Delta^m}_{PL}(X \times \Delta^m).$$

Let us regard n as fixed for the remainder of this lecture. We will prove Proposition 3 by showing that the map of geometric realizations

$$| \varinjlim_{T \subseteq (0,1)} S_n \operatorname{Shv}^{\Delta^\bullet, T}_{PL}(X \times \Delta^\bullet)| \to |S_n \operatorname{Shv}^{\Delta^\bullet}_{PL}(X \times \Delta^\bullet)|.$$

is a homotopy equivalence. Note that for fixed m and $T \subseteq (0,1)$, the map

$$S_n \operatorname{Shv}^{\Delta^m, T}_{PL}(X \times \Delta^m) \to S_n \operatorname{Shv}^{\Delta^m}_{PL}(X \times \Delta^m)$$

is the inclusion of a summand: the target space classifies diagrams

$$\mathcal{F}_1 \to \cdots \to \mathcal{F}_n$$

in $\operatorname{Shv}^{\Delta^m}_{PL}(X \times \Delta^m)$, and the domain classifies diagrams which have the additional property that each $\mathcal{F}_i$ is transverse to each $t \in T$. If the diagram $\mathcal{F}_1 \to \cdots \to \mathcal{F}_n$ is fixed, then the collection of those $t \in (0,1)$ such that each $\mathcal{F}_i$ is transverse to T comprise a subset $U \subseteq (0,1)$, and the homotopy fiber of the map θ_m over the point corresponding to $\mathcal{F}_1 \to \cdots \to \mathcal{F}_n$ can be identified with the nerve of the category of nonempty finite subsets of U. This category is either empty (if $U = \emptyset$) or contractible (if $U \neq \emptyset$). It follows that each of the maps θ_m exhibits $\varinjlim_{T \subseteq (0,1)} S_n \operatorname{Shv}^{\Delta^m, T}_{PL}(X \times \Delta^m)$ as a summand of $S_n \operatorname{Shv}^{\Delta^m}_{PL}(X \times \Delta^m)$: namely, the summand consisting of those diagrams $\mathcal{F}_1 \to \cdots \to \mathcal{F}_n$ which are all transverse to *some* $t \in (0,1)$.

Construction 4. Let $Z_\bullet$ be a simplicial space. For every simplicial set K, we let $Z_\bullet(K)$ denote the homotopy inverse limit $\varprojlim_{\Delta^p \to K} Z_p$.

We define a new simplicial space $\operatorname{Ex}(Z_\bullet)$ by the formula

$$\operatorname{Ex}(Z_\bullet)_q = Z_\bullet(\operatorname{Sd} \Delta^q).$$

Note that the "last vertex maps" $\operatorname{Sd} \Delta^q \to \Delta^q$ are compatible as q varies and give rise to a map of simplicial spaces $\rho_{Z_\bullet} : Z_\bullet \to \operatorname{Ex}(Z_\bullet)$.

Lemma 5. *For any simplicial space $Z_\bullet$, the map $\rho_{Z_\bullet} : Z_\bullet \to \operatorname{Ex}(Z_\bullet)$ induces a homotopy equivalence of geometric realizations*

$$|Z_\bullet| \to |\operatorname{Ex}(Z_\bullet)|.$$

Sketch. If $Z_\bullet$ is a simplicial set (meaning that each Z_n is discrete), then this is classical. The general case can be reduced to this: the ∞-category of simplicial spaces is the underlying ∞-category of the model category of *bisimplicial sets*, equipped with the Reedy model structure. Moreover, if $Z_{\bullet\bullet}$ is a bisimplicial set which is Reedy fibrant, then the Ex of Construction 4 can be computed by levelwise application of the usual Ex functor on simplicial sets. □

Let us now specialize to the case where $Z_\bullet$ is the simplicial space $S_n \operatorname{Shv}_{PL}^{\Delta^\bullet}(X \times \Delta^\bullet)$. In this case, we can identify $\operatorname{Ex}(Z_\bullet)$ with the simplicial space whose m-simplices are given by $S_n \operatorname{Shv}_{PL}^{|\operatorname{Sd}\Delta^m|}(X \times |\operatorname{Sd}\Delta^m|)$. Recall that there is a canonical piecewise linear homeomorphism of $|\operatorname{Sd}\Delta^m|$ with $|\Delta^m|$, which is functorial for *injective* maps between simplices. These homeomorphisms determine an equivalence $\theta_{Z_\bullet}$ between the underlying semisimplicial spaces of $Z_\bullet$ and $\operatorname{Ex}(Z_\bullet)$, which we will denote by $Z_\bullet^s$ and $\operatorname{Ex}(Z_\bullet)^s$.

Let $Y_\bullet$ denote the simplicial subspace of $Z_\bullet$ given by $\varinjlim_{T \subseteq (0,1)} S_n \operatorname{Shv}_{PL}^{\Delta^\bullet, T}(X \times \Delta^\bullet)$. The map $\theta_{Z_\bullet}$ restricts to a morphism of underlying semisimplicial spaces

$$\theta_{Y_\bullet} : Y_\bullet^s \to \operatorname{Ex}(Y_\bullet)^s$$

(which is now *not* a levelwise homotopy equivalence). This is not identical to the map of semisimplicial spaces $\rho_{Y_\bullet}$ appearing in Lemma 5. However, they differ by a simplicial homotopy and therefore induce the same map after passing to geometric realizations. It follows from Lemma 5 that $\theta_{Y_\bullet}$ induces a homotopy equivalence

$$|Y_\bullet^s| \to |\operatorname{Ex}(Y_\bullet)^s|.$$

We wish to show that the inclusion $i_\bullet : Y_\bullet \hookrightarrow Z_\bullet$ induces a homotopy equivalence of geometric realizations. For each integer $p \geq 0$, let $\operatorname{Ex}^p(Y_\bullet)$ denote the result of p-fold application of the functor Ex to the simplicial space $Y_\bullet$, and define $\operatorname{Ex}^p(Z_\bullet)$ similarly. Using the above arguments we can identify $|i_\bullet|$ with the induced of geometric realizations

$$|\varinjlim \operatorname{Ex}^p(Y_\bullet)^s| \to |\varinjlim \operatorname{Ex}^p(Z_\bullet)^s|.$$

It will therefore suffice to prove the following:

Proposition 6. *For each integer m, the canonical map $\varinjlim \operatorname{Ex}^p(Y_\bullet)_m \to \varinjlim \operatorname{Ex}^p(Z_\bullet)_m$ is a homotopy equivalence.*

Unwinding the definitions, Proposition 6 asserts that for every diagram $\mathcal{F}_1 \to \cdots \to \mathcal{F}_n$ in $\operatorname{Shv}_{PL}^{\Delta^m}(X \times \Delta^m)$, there exists an integer $p \geq 0$ such that on each simplex σ of the p-fold barycentric subdivision of Δ^m, there exists a real number t_σ which is transverse to each $\mathcal{F}_i|_{X \times \sigma}$.

Choose compatible triangulations Σ of $X_{01} \times [0,1] \times \Delta^m$, Σ' of $[0,1] \times \Delta^m$, and Σ'' of Δ^m, such that each $\mathcal{G}_i$ is constructible with respect to Σ. Let $V \subseteq [0,1] \times \Delta^m$ be the union of the interiors of those simplices $\sigma \in \Sigma'$ for which the map $\sigma \to \Delta^m$ is not injective. It is easy to see that $V \subseteq (0,1) \times \Delta^m$ and that the projection map $V \to \Delta^m$ is surjective. Since V is open, there exists an open cover U_α of Δ^m and a collection of real numbers $t_\alpha \in (0,1)$ such that each product $\{t_\alpha\} \times U_\alpha$ is contained in V. Choose $p \gg 0$ so that each simplex σ of the p-fold barycentric subdivision of Δ^m is contained in some U_α. We claim that p satisfies our requirements: more precisely, each restriction $\mathcal{F}_i|_{X \times \sigma}$ is transverse to t_α. To prove this, it will suffice to show that $\mathcal{F}_i|_{X_{01} \times [0,1] \times \Delta^m}$ is ULA over $[0,1] \times \Delta^m$ over the open set V.

Let

$$f : X_{01} \times [0,1] \times \Delta^m \to [0,1] \times \Delta^m$$

and

$$g : [0,1] \times \Delta^m \to \Delta^m$$

be the projection maps, let $\theta_0 \in \Sigma$, let $\sigma_0 = f(\theta_0) \in \Sigma'$, and let $\sigma \in \Sigma'$ be a simplex containing σ_0. Set $\tau_0 = g(\sigma_0) \in \Sigma''$ and $\tau = g(\sigma) \in \Sigma''$. We wish to show that if the interior of σ_0 is contained in V, then the canonical map

$$\mathcal{F}_i(\theta_0) \to \varprojlim_{\theta_0 \subseteq \theta, f(\theta) = \sigma} \mathcal{F}_i(\theta)$$

is an equivalence. Our assumption that the interior of σ_0 is contained in V guarantees that σ is the *unique* simplex of Σ' which contains σ_0 and whose image is τ, so we can rewrite our map as

$$\mathcal{F}_i(\theta_0) \to \varprojlim_{\theta_0 \subseteq \theta, (g \circ f)(\theta) = \tau} \mathcal{F}_i(\theta).$$

This map is an equivalence by virtue of our assumption that $\mathcal{F}_i$ is ULA over Δ^m.