The Assembly Map, Part II (Lecture 25)

November 3, 2014

Let X be a finite polyhedron. In the previous lecture, we introduced an infinite loop space $K_\Delta(X)$, which is given by the geometric realization of the simplicial space

$$[n] \mapsto K(\mathrm{Shv}^{\Delta^n}_{PL}(X \times \Delta^n)).$$

Moreover, we constructed a map of connective spectra

$$\Omega^{-\infty} K_\Delta(X) \to A(X).$$

Our goal in this lecture and the next is to prove the following:

Theorem 1. *The map $K_\Delta(X) \to \Omega^\infty A(X)$ is a model for the assembly map in A-theory. In particular, there is a canonical homotopy equivalence $K_\Delta(X) \simeq \Omega^\infty(X_+ \wedge A(*))$.*

As a first step, we consider functoriality in X. Recall that for any map of finite polyhedra $f : X \to Y$, the pushforward map

$$(f \times \mathrm{id})_* : \mathrm{Shv}(X \times S) \to \mathrm{Shv}(Y \times S)$$

preserves the property of being ULA over S. It follows that we obtain a map of simplicial ∞-categories

$$\mathrm{Shv}^{\Delta^\bullet}_{PL}(X \times \Delta^\bullet) \to \mathrm{Shv}^{\Delta^\bullet}_{PL}(Y \times \Delta^\bullet).$$

Taking K-theory and passing to geometric realizations, we obtain a map of infinite loop spaces $K_\Delta(X) \to K_\Delta(Y)$. In other words, we can regard K_Δ as a functor from the ordinary category Poly of finite polyhedra (with morphisms given by PL maps) to the ∞-category of spaces (or even of E_∞-spaces).

The category Poly has a canonical simplicial enrichment: to every pair of finite polyhedra X and Y, we can associate a Kan complex $\mathrm{Map}(X, Y)$ whose n-simplices are given by piecewise linear maps from $X \times \Delta^n$ into Y. As a simplicially enriched category, Poly is weakly equivalent to the simplicially enriched category of finite CW complexes. This follows from two observations:

- Every finite CW complex is homotopy equivalent to a finite polyhedron.
- For every pair of finite polyhedra X and Y, the Kan complex $\mathrm{Map}(X, Y)$ is homotopy equivalent to the singular simplicial set of the topological space Y^X of all continuous maps from X into Y (more informally: there are no obstructions to approximating arbitrary continuous maps between polyhedra by piecewise-linear maps).

Using this fact, it is not difficult to show that the ∞-category $\mathcal{S}^{\mathrm{fin}}$ of finite spaces can be obtained from the ordinary category Poly by formally inverting all maps of the form $X \times \Delta^n \to X$. In fact, it suffices to consider the case $n = 1$ (since the n-simplex Δ^n is a retract of a product of copies of Δ^1). In other words, we have the following:

Claim 2. *Let $\mathcal{C}$ be an ∞-category. Then composition with the canonical map* $\mathrm{Poly} \to \mathcal{S}^{\mathrm{fin}}$ *induces a fully faithful embedding*

$$\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Poly}, \mathcal{C}),$$

whose essential image is spanned by the collection of those functors $F : \mathrm{Poly} \to \mathcal{C}$ *with the property that for any finite polyhedron X, the induced map $F(X \times \Delta^1) \to F(X)$ is an equivalence in $\mathcal{C}$.*

We would like to apply Claim 2 to the functor $X \mapsto K_{\Delta}(X)$.

Proposition 3. *For any finite polyhedron X, the canonical map $K_{\Delta}(X \times \Delta^1) \to K_{\Delta}(X)$ is a homotopy equivalence.*

The map of Proposition 3 has a right homotopy inverse, induced by the inclusion $X \times \{0\} \hookrightarrow X \times \Delta^1$. To check that this map is a left homotopy inverse, it will suffice to establish the following:

Lemma 4. *Let $f, g : X \to Y$ be homotopic maps of finite polyhedra. Then f and g induce homotopic maps $f_* g_* : K_{\Delta}(X) \to K_{\Delta}(Y)$.*

Proof. It suffices to prove Lemma 4 in the "universal" case where $Y = X \times \Delta^1$ and f and g are the two inclusions $X \times \{i\} \hookrightarrow X \times \Delta^1$. Let $\mathcal{J}$ denote the slice category $\mathbf{\Delta}_{/[1]}$ of nonempty finite linearly ordered sets $[n]$ equipped with a map $[n] \to [1]$. Then $\mathcal{J}$ contains full subcategories $\mathcal{J}_0, \mathcal{J}_1 \subseteq \mathcal{J}$, spanned by those objects of the form $[n] \to \{0\} \subseteq [1]$ and $[n] \to \{1\} \subseteq [1]$, respectively. Each of these subcategories is equivalent to $\mathbf{\Delta}$. To each object $[n] \to [1]$ in $\mathcal{J}$, we can associate a map of finite polyhedra

$$X \times \Delta^n \to X \times \Delta^1 \times \Delta^n,$$

which induces a pushforward functor

$$\mathrm{Shv}^{\Delta^n}_{PL}(X \times \Delta^n) \to \mathrm{Shv}^{\Delta^n}_{PL}(X \times \Delta^1 \times \Delta^n).$$

Taking K-theory and passing to the colimit, we obtain a map

$$\varinjlim_{[n] \in \mathcal{J}} K(\mathrm{Shv}^{\Delta^n}_{PL}(X \times \Delta^n)) \to K_{\Delta}(X \times \Delta^1).$$

Note that the composition of this map with the natural maps

$$\varinjlim_{[n] \in \mathcal{J}_0} K(\mathrm{Shv}^{\Delta^n}_{PL}(X \times \Delta^n)) \to \varinjlim_{[n] \in \mathcal{J}} K(\mathrm{Shv}^{\Delta^n}_{PL}(X \times \Delta^n))$$

$$\varinjlim_{[n] \in \mathcal{J}_1} K(\mathrm{Shv}^{\Delta^n}_{PL}(X \times \Delta^n)) \to \varinjlim_{[n] \in \mathcal{J}} K(\mathrm{Shv}^{\Delta^n}_{PL}(X \times \Delta^n))$$

coincide with f_* and g_*, respectively. It will therefore suffice to show that these latter maps are homotopy equivalences. Both have left homotopy inverses induced by the map

$$\varinjlim_{[n] \in \mathcal{J}} K(\mathrm{Shv}^{\Delta^n}_{PL}(X \times \Delta^n)) \to K_{\Delta}(X)$$

determined by the forgetful functor $\pi : \mathcal{J} \to \mathbf{\Delta}$. To show that this map is a homotopy equivalence, it will suffice to show that π is right cofinal. In other words, it will suffice to show that for each object $[n] \in \mathbf{\Delta}$, the category

$$\mathcal{J} \times_{\mathbf{\Delta}} \mathbf{\Delta}_{/[n]} = \mathbf{\Delta}_{/[1]} \times_{\mathbf{\Delta}} \mathbf{\Delta}_{/[n]}$$

is weakly contractible. This is clear, since it is the category of simplices of the weakly contractible simplicial set $\Delta^1 \times \Delta^n$. $\square$

It follows from Proposition 3 and Claim 2 that we can regard the construction $X \mapsto K_\Delta(X)$ as a functor from the ∞-category $\mathcal{S}^{\text{fin}}$ of finite spaces to the ∞-category of E_∞ spaces. Moreover, since the map

$$\Omega^{-\infty} K_\Delta(X) \to A(X)$$

constructed in the previous lecture was functorial for maps of finite polyhedra, it can be regarded as a natural transformation between functors from $\mathcal{S}^{\text{fin}}$ to spectra. We next make a simple observation:

Proposition 5. *The map $\Omega^{-\infty} K_\Delta(X) \to A(X)$ is an equivalence when X is a point.*

Proof. When X is a point, a constructible sheaf $\mathcal{F}$ on $X \times \Delta^n$ is ULA over Δ^n if and only if it is locally constant. Consequently, we can identify $\text{Shv}_{PL}^{\Delta^\bullet}(X \times \Delta^\bullet)$ with the constant simplicial ∞-category taking the value Sp^{fin}. It follows that $K_\Delta(X)$ can be identified with $K(\text{Sp}^{\text{fin}}) \simeq \Omega^\infty A(X)$. $\square$

It follows from Proposition 5 that the colimit-preserving approximations to the functors $\Omega^{-\infty} K_\Delta(X)$ and $A(X)$ are the same. In other words, for any finite space X we have a commutative diagram

$$\begin{array}{ccc} & X_+ \wedge A(*) & \\ {}^{\theta_X}\swarrow & & \searrow \\ \Omega^{-\infty} K_\Delta(X) & \longrightarrow & A(X). \end{array}$$

We can now formulate a more precise version of Theorem 1: the map θ_X is a homotopy equivalence of spectra. To prove this, we note that the collection of those spaces X for which θ_X is a homotopy equivalence contains the one-point space (by Proposition 5) and the empty space (since the domain and codomain of θ_X both vanish in this case). Consequently, to show that it contains all finite spaces, it will suffice to show that it is closed under (homotopy) pushouts. Since the functor $X \mapsto X_+ \wedge A(*)$ preserves pushout squares, we are reduced to the following:

Proposition 6. *Suppose we are given a pushout diagram*

$$\begin{array}{ccc} X_{01} & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X \end{array}$$

in the ∞-category of finite spaces. Then the diagram of spectra

$$\begin{array}{ccc} \Omega^{-\infty} K_\Delta(X_{01}) & \longrightarrow & \Omega^{-\infty} K_\Delta(X_0) \\ \downarrow & & \downarrow \\ \Omega^{-\infty} K_\Delta(X_1) & \longrightarrow & \Omega^{-\infty} K(X) \end{array}$$

is also a homotopy pushout square.

In the statement of Proposition 6, we may assume without loss of generality that X_{01}, X_0, and X_1 are represented by finite polyhedra, and that the maps $X_{01} \to X_0$ and $X_{01} \to X_1$ are given by embeddings of finite polyhedra. We will assume that X is given by the homotopy pushout

$$X_0 \amalg_{X_{01} \times \{0\}} (X_{01} \times [0,1]) \amalg_{X_{01} \times \{1\}} X_1.$$

For each real number t with $0 < t < 1$, let X_t denote the subcomplex of X given by $X_{01} \times \{t\}$. We will say that a sheaf $\mathcal{F} \in \text{Shv}_{PL}^S(X \times S)$ is *transverse* to X_t if the restriction $\mathcal{F}|_{X_t \times S}$ is ULA over S. Let $\text{Shv}_{PL}^{S,t}(X \times S)$ denote the full subcategory of $\text{Shv}_{PL}^S(X \times S)$ spanned by those sheaves which are transverse to t. As S ranges over all simplices, we obtain a simplicial ∞-category $\mathrm{Shv}_{PL}^{\Delta^\bullet}(X \times \Delta^\bullet)$. Let $K_\Delta^t(X)$ denote the geometric realization of the simplicial space

$$K(\mathrm{Shv}_{PL}^{\Delta^\bullet,t}(X \times \Delta^\bullet)).$$

Let us compute $K_\Delta^t(X)$.

Let S be a finite polyhedron, and let $\mathcal{C}_-^S$ and $\mathcal{C}_+^S$ denote the full subcategories of $\mathrm{Shv}_{PL}(X_{\leq t} \times S)$ and $\mathrm{Shv}_{PL}(X_{\geq t} \times S)$ spanned by those sheaves which are ULA over S and which vanish when restricted to X_t. Note that $X_{\leq t}$ and $X_{\geq t}$ contain X_0 and X_1 as deformation retracts. Using a slight variant of the proof of Lemma 4, one can show that the canonical maps

$$K_\Delta(X_0) \to |\,\mathcal{C}_-^{\Delta^\bullet}\,|$$

$$K_\Delta(X_1) \to |\,\mathcal{C}_+^{\Delta^\bullet}\,|$$

are homotopy equivalences.

Note that if $\mathcal{F} \in \mathrm{Shv}_{PL}^{S,T}(X \times S)$, then we have a canonical fiber sequence

$$\mathcal{F}' \to \mathcal{F} \to i_* i^* \mathcal{F}.$$

Since $\mathcal{F}$ and $i^* \mathcal{F}$ are ULA over S, it follows that $i_* i^* \mathcal{F}$ and $\mathcal{F}'$ are ULA over S. Because $\mathcal{F}'\,|_{X_t}$ vanishes, we can write $\mathcal{F}'$ as a direct sum $\mathcal{F}'_- \oplus \mathcal{F}'_+$, where $\mathcal{F}'_- \in \mathcal{C}_-^S$ and $\mathcal{F}'_+ \in \mathcal{C}_+^S$. The construction $\mathcal{F} \mapsto (\mathcal{F}'_-, \mathcal{F}'_+, i^* \mathcal{F})$ determines an exact functor

$$\mathrm{Shv}_{PL}^{S,T}(X \times S) \to \mathcal{C}_-^S \times \mathcal{C}_+^S \times \mathrm{Shv}_{PL}^S(X_t \times S).$$

This map has a right homotopy inverse (given by pushing forward to $X \times S$ and forming the direct sum). It follows from the additivity theorem that this right homotopy inverse is actually a two-sided homotopy inverse after passing to K-theory. In particular, we obtain a homotopy equivalence

$$K(\mathrm{Shv}_{PL}^{S,t}(X \times S)) \simeq K(\mathcal{C}_-^S) \times K(\mathcal{C}_+^S) \times K(\mathrm{Shv}_{PL}^S(X_t \times S)).$$

Taking S to range over simplices and passing to geometric realizations, we obtain an equivalence

$$K_\Delta^t(X) \simeq K_\Delta(X_0) \times K_\Delta(X_1) \times K_\Delta(X_t).$$

We will elaborate more on this in the next lecture.