The Additivity Theorem (Lecture 17)

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Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits. In the previous lecture, we introduced an infinite loop space $K(\mathcal{C})$, the *Waldhausen K-theory space* of $\mathcal{C}$. Note that the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$ of arrows in $\mathcal{C}$ satisfies the same hypotheses as $\mathcal{C}$, so we can also consider the K-theory space $K(\mathrm{Fun}(\Delta^1, \mathcal{C}))$. Our goal in this lecture is to prove the following fundamental result:

Theorem 1 (Additivity Theorem)**.** *The functor*

$$F : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C} \times \mathcal{C}$$

$$F(\alpha : C \to D) = (C, \mathrm{cofib}(\alpha))$$

induces a homotopy equivalence

$$K(\mathrm{Fun}(\Delta^1, \mathcal{C})) \to K(\mathcal{C} \times \mathcal{C}) \simeq K(\mathcal{C}) \times K(\mathcal{C}).$$

Before giving the proof of Theorem 1, let us describe some of its consequences. We first note that the functor $F : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ admits a right homotopy inverse, given by the construction

$$(C, D) \mapsto (C \to C \vee D).$$

We therefore obtain:

Corollary 2. *The functor $(C, D) \mapsto (C \to C \vee D)$ induces a homotopy equivalence*

$$K(\mathcal{C} \times \mathcal{C}) \to K(\mathrm{Fun}(\Delta^1, \mathcal{C})).$$

To state the next Corollary, we will need a bit of notation. Suppose that $\mathcal{C}$ and $\mathcal{D}$ are pointed ∞-categories which admit finite colimits, and that $G : \mathcal{C} \to \mathcal{D}$ is a functor which preserves finite colimits. We let G_* denote the associated map (of infinite loop spaces) from $K_0(\mathcal{C})$ to $K_0(\mathcal{D})$.

Corollary 3. *Let $G', G, G'' : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ be the functors given by*

$$G'(C \to D) = C \qquad G(C \to D) = D \qquad G''(C \to D) = D/C.$$

Then we have $G_* = G'_* + G''_*$ (in the abelian group of homotopy classes of maps from $K_0(\mathrm{Fun}(\Delta^1, \mathcal{C}))$ to $K_0(\mathcal{C})$).

Proof. By virtue of Corollary 2, it will suffice to show that the corresponding equality holds in the space of maps from $K(\mathcal{C} \times \mathcal{C})$ to $K(\mathcal{C})$, which follows immediately from the definition of the addition law on $K(\mathcal{C})$. ☐

Corollary 4. *Let $\mathcal{C}$ and $\mathcal{D}$ be pointed ∞-categories which admit finite colimits, and suppose we are given a cofiber sequence*

$$F' \xrightarrow{\alpha} F \to F''$$

of functors from $\mathcal{D}$ to $\mathcal{C}$ which preserve finite colimits. Then $F_* = F'_* + F''_*$.

Proof. The natural transformation α determines a functor $H : \mathcal{D} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ which preserves finite colimits. Unwinding the definitions, we have

$$F'_* = G'_* H_* \quad F_* = G_* H_* \quad F''_* = G''_* H_*$$

where $G', G, G'' : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ are defined as in Corollary 3. The desired result now follows from the equality $G_* = G'_* + G''_*$. □

Corollary 5. *Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits. Then the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ induces the map $K(\mathcal{C}) \to K(\mathcal{C})$ given by multiplication by -1.*

Proof. Apply Corollary 4 to the cofiber sequence of functors

$$\mathrm{id} \to * \to \Sigma.$$

□

Corollary 6. *Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits. Then the canonical map $\mathcal{C} \to SW(\mathcal{C})$ induces a homotopy equivalence $K(\mathcal{C}) \to K(SW(\mathcal{C}))$*

Proof. The K-theory space of $SW(\mathcal{C})$ can be identified with the direct limit of the sequence

$$K(\mathcal{C}) \stackrel{\Sigma_*}{\to} K(\mathcal{C}) \stackrel{\Sigma_*}{\to} K(\mathcal{C}) \to \cdots$$

□

It follows from Corollary 6 that as far as Waldhausen K-theory is concerned, we might as well always be working with stable ∞-categories (which will be the focus of our attention in the next several lectures).

Let us now turn to the proof of Theorem 1. We have an evident evaluation functor

$$e : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$$
$$e(C \to D) = C.$$

Fix a zero object $* \in \mathcal{C}$. The fiber of e over $*$ can be identified with the ∞-category $\mathcal{C}_*$ of pointed objects of $\mathcal{C}$ (that is, objects $D \in \mathcal{C}$ equipped with a map $* \to D$). Since $\mathcal{C}$ is pointed, this ∞-category is equivalent to $\mathcal{C}$ itself. We therefore have a fiber sequence of ∞-categories,

$$\mathcal{C} \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \stackrel{e}{\to} \mathcal{C}.$$

which gives a diagram of K-theory spaces

$$K(\mathcal{C}) \to K(\mathrm{Fun}(\Delta^1, \mathcal{C})) \stackrel{e_*}{\to} K(\mathcal{C}).$$

Theorem 1 is equivalent to the assertion that this diagram is again a fiber sequence.

It follows immediately from the definitions that for every integer $n \geq 0$, we have a fiber sequence of ∞-categories

$$\mathrm{Gap}_{[n]}(\mathcal{C}) \to \mathrm{Gap}_{[n]}(\mathrm{Fun}(\Delta^1, \mathcal{C})) \to \mathrm{Gap}_{[n]}(\mathcal{C}).$$

Passing to underlying Kan complexes and allowing n to vary, we obtain a fiber sequence of simplicial spaces

$$S_\bullet(\mathcal{C}) \to S_\bullet(\mathrm{Fun}(\Delta^1, \mathcal{C})) \to S_\bullet(\mathcal{C}).$$

We would like to show that the induced diagram of geometric realizations remains a fiber sequence. This is not obvious: in general, a fiber sequence of simplicial pointed spaces

$$X_\bullet \to Y_\bullet \to Z_\bullet$$

need not yield a fiber sequence $|X_\bullet| \to |Y_\bullet| \to |Z_\bullet|$. One can show that this holds whenever each of the spaces Z_n is connected, but this hypothesis is far too strong for our situation (remember that the connected components of $S_1(\mathcal{C})$ can be identified with the equivalence classes of objects in $\mathcal{C}$; in particular, $S_1(\mathcal{C})$ is never connected unless $\mathcal{C}$ is trivial). We will instead use the following criterion:

Proposition 7. *Let $\mathcal{S}$ denote the ∞-category of spaces and let $\mathcal{J}$ be a small category (or ∞-category). Suppose we are given a natural transformation $X \to Y$ of functors from $\mathcal{J}$ to $\mathcal{S}$ which satisfies the following condition:*

- $(*)$ *For every morphism $I \to J$ in $\mathcal{J}$, the associated map*

$$\varinjlim_{J \to K} X(K) \times_{Y(K)} Y(I) \to \varinjlim_{I \to K} X(K) \times_{Y(K)} Y(I)$$

  is a homotopy equivalence.

Suppose we have a pullback diagram of functors

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y. \end{array}$$

Then:

- (a) *The map $X' \to Y'$ also satisfies $(*)$.*
- (b) *The diagram*

$$\begin{array}{ccc} \varinjlim X' & \longrightarrow & \varinjlim X \\ \downarrow & & \downarrow \\ \varinjlim Y' & \longrightarrow & \varinjlim Y \end{array}$$

  is also a pullback square.

Proof. We first prove (a). Fix a morphism $I \to J$ in $\mathcal{J}$. We have a commutative diagram

$$\begin{array}{ccc} \varinjlim_{J \to K} X'(K) \times_{Y'(K)} Y'(I) & \longrightarrow & \varinjlim_{I \to K} X'(K) \times_{Y'(K)} Y'(I) \\ \downarrow & & \downarrow \\ \varinjlim_{J \to K} X'(K) \times_{Y'(K)} Y(I) & \longrightarrow & \varinjlim_{I \to K} X'(K) \times_{Y'(K)} Y(I) \\ \downarrow & & \downarrow \\ \varinjlim_{J \to K} X(K) \times_{Y(K)} Y(I) & \longrightarrow & \varinjlim_{I \to K} X(K) \times_{Y(K)} Y(I) \end{array}$$

where each square is a pullback. Since the lower horizontal map is a homotopy equivalence, so is the upper horizontal map.

For every morphism $I \to J$ in $\mathcal{J}$, let $F(I \to J)$ denote the colimit $\varinjlim_{J \to K} X(K) \times_{Y(K)} Y(I)$. *A priori*, the functor F is covariant in I and contravariant in J. However, condition $(*)$ implies that F is actually independent of J. More precisely, we can write $F(I \to J) = F_0(I)$ for some functor $F_0 : \mathcal{J} \to \mathcal{S}$. Abstractly, we can describe F_0 as the left Kan extension of F along the forgetful functor $(I \to J) \mapsto I$ (from the twisted arrow category of $\mathcal{J}$ to $\mathcal{J}$). Concretely, we can write $F_0(I) = F(I \to I) = \varinjlim_{I \to K} X(K) \times_{Y(K)} Y(I)$. Note that we have an evident projection map $F_0(I) \to Y(I)$, and condition $(*)$ implies that for every map $I \to J$, the associated map

$$\begin{array}{ccc} F_0(I) & \longrightarrow & F_0(J) \\ \downarrow & & \downarrow \\ Y(I) & \longrightarrow & Y(J) \end{array}$$

is a pullback square. It follows that for each $I \in \mathcal{I}$, the diagram

$$\begin{array}{ccc} F_0(I) & \longrightarrow & \varinjlim F_0 \\ \downarrow & & \downarrow \\ Y(I) & \longrightarrow & \varinjlim Y \end{array}$$

is a pullback square.

Using (a), we can apply the same reasoning to the natural transformation $X' \to Y'$ to obtain a functor $F'_0 : \mathcal{J} \to \mathcal{S}$. The proof of (a) shows that for each $I \in \mathcal{I}$, the diagram

$$\begin{array}{ccc} F'_0(I) & \longrightarrow & F_0(I) \\ \downarrow & & \downarrow \\ Y'(I) & \longrightarrow & Y(I) \end{array}$$

is a pullback square. It follows that we also have a pullback square

$$\begin{array}{ccc} F'_0(I) & \longrightarrow & \varinjlim F_0 \\ \downarrow & & \downarrow \\ Y'(I) & \longrightarrow & \varinjlim Y. \end{array}$$

Passing to the colimit over I (and using the fact that colimits in $\mathcal{S}$ commute with base change), we obtain a pullback square

$$\begin{array}{ccc} \varinjlim F'_0 & \longrightarrow & \varinjlim F_0 \\ \downarrow & & \downarrow \\ \varinjlim Y' & \longrightarrow & \varinjlim Y. \end{array}$$

We conclude by observing that there are canonical equivalences

$$\begin{aligned} \varinjlim F_0 &\simeq \varinjlim F \\ &\simeq \varinjlim_{I \to J} \varinjlim_{J \to K} X(K) \times_{Y(K)} Y(I) \\ &\simeq \varinjlim_{K} X(K) \times_{Y(K)} \varinjlim_{I \to J \to K} Y(I) \\ &\simeq \varinjlim_{K} X(K) \times_{Y(K)} Y(K) \\ &\simeq \varinjlim X, \end{aligned}$$

and similarly $\varinjlim F'_0 \simeq \varinjlim X'$. □

Exercise 8. Let $X \to Y$ be a natural transformation of functors $\mathcal{J} \to \mathcal{S}$. Let us say that a morphism $f : I \to J$ in $\mathcal{J}$ is *good* if the natural map

$$\varinjlim_{J \to K} X(K) \times_{Y(K)} Y(I) \to \varinjlim_{I \to K} X(K) \times_{Y(K)} Y(I)$$

is a homotopy equivalence. Show that if we are given a pair of morphisms

$$I \xrightarrow{f} J \xrightarrow{g} K$$

in $\mathcal{J}$ such that g is good, then f is good if and only if the composition $g \circ f$ is good.

In order to prove Theorem 1, it will suffice to show that the evaluation map e induces a map of simplicial spaces $S_\bullet(\mathrm{Fun}(\Delta^1, \mathcal{C})) \to S_\bullet(\mathcal{C})$ which satisfies the requirements of Proposition 7 (where we take $\mathcal{J} = \mathbf{\Delta}^{\mathrm{op}}$ to be the opposite of the category of nonempty finite linearly ordered sets). Fix a map of linearly ordered sets $\alpha : [n'] \to [n]$; we wish to show that the induced map

$$\varinjlim_{\beta:[m]\to[n']} S_m(\mathrm{Fun}(\Delta^1, \mathcal{C})) \times_{S_m(\mathcal{C})} S_n(\mathcal{C}) \to \varinjlim_{\beta:[m]\to[n]} S_m(\mathrm{Fun}(\Delta^1, \mathcal{C})) \times_{S_m(\mathcal{C})} S_n(\mathcal{C})$$

is a homotopy equivalence. Using Exercise 8, we may reduce to the case where $n' = 0$.

Let us fix a point of $S_n(\mathcal{C})$, corresponding to an $[n]$-gapped object $X' \in \mathcal{C}$. Passing to the homotopy fibers over X', we are reduced to proving that the map

$$\theta : \varinjlim_{\beta:[m]\to[0]} S_m(\mathrm{Fun}(\Delta^1, \mathcal{C})) \times_{S_m(\mathcal{C})} \{X'\} \to \varinjlim_{\beta:[m]\to[n]} S_m(\mathrm{Fun}(\Delta^1, \mathcal{C})) \times_{S_m(\mathcal{C})} \{X'\}$$

is a homotopy equivalence. Let $\mathrm{cofib} : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ be the functor given by $(f : C \to D) \mapsto \mathrm{cofib}(f)$, so that cofib induces maps $S_m(\mathrm{Fun}(\Delta^1, \mathcal{C})) \to S_m(\mathcal{C})$. It follows immediately from the definitions that the composite map

$$\varinjlim_{\beta:[m]\to[0]} S_m(\mathrm{Fun}(\Delta^1, \mathcal{C})) \times_{S_m(\mathcal{C})} \{X'\} \xrightarrow{\theta} \varinjlim_{\beta:[m]\to[n]} S_m(\mathrm{Fun}(\Delta^1, \mathcal{C})) \times_{S_m(\mathcal{C})} \{X'\} \xrightarrow{\theta'} \varinjlim_{[m]\in\mathbf{\Delta}^{\mathrm{op}}} S_m(\mathcal{C})$$

is a homotopy equivalence, so that θ is a homotopy equivalence if and only if θ' is a homotopy equivalence. In particular, we see that the condition that θ is a homotopy equivalence is independent of the choice of map $\alpha : [0] \to [n]$. We may therefore assume without loss of generality that $\alpha(0) = n$.

For every map $\beta : [m] \to [n]$, let X'_β denote the image of X' in $\mathrm{Gap}_{[m]}(\mathcal{C})$. Unwinding the definitions, we can identify $S_m(\mathrm{Fun}(\Delta^1, \mathcal{C})) \times_{S_m(\mathcal{C})} \{X'\}$ with the Kan complex

$$Z_\beta = \mathrm{Gap}_{[m]}(\mathcal{C})^{\simeq}_{X'_\beta/},$$

whose vertices are maps $X'_\beta \to X$ in the ∞-category of $[m]$-gapped objects of $\mathcal{C}$. Note that if $\beta \leq \beta'$, then there is a canonical map $\tau_{\beta,\beta'} : Z_\beta \to Z_{\beta'}$ given by $X \mapsto X'_{\beta'} \amalg_{X'_\beta} X$.

Let us regard $[n]$ as fixed, and let $\mathcal{I}$ denote the category whose objects are nonempty finite linearly ordered sets $[m]$ and monotone maps $\beta : [m] \to [n]$. Let $\mathcal{I}_0 \subseteq \mathcal{I}$ be the full subcategory consisting of those maps $\beta : [m] \to [n]$ which take the constant value n (so that $\mathcal{I}_0$ is equivalent to the category $\mathbf{\Delta}$). The construction $\beta \mapsto Z_\beta$ determines a functor $\mathcal{I}^{\mathrm{op}} \to \mathcal{S}$, and we wish to show that the canonical map

$$\varinjlim_{\beta\in\mathcal{I}_0^{\mathrm{op}}} Z_\beta \to \varinjlim_{\beta\in\mathcal{I}^{\mathrm{op}}} Z_\beta$$

is a homotopy equivalence.

The key observation is that the construction $\beta \mapsto Z_\beta$ has a little bit of extra functoriality. Let us define an enlargement $\mathcal{I}_+$ of $\mathcal{I}$ as follows:

- The objects of $\mathcal{I}_+$ are nonempty finite linearly ordered sets $[m]$ equipped with monotone maps $\beta : [m] \to [n]$.
- A morphism from $\beta : [m] \to [n]$ to $\beta' : [m'] \to [n]$ in $\mathcal{I}_+$ consists of a monotone map $\gamma : [m] \to [m']$ such that $\beta(i) \geq \beta'(\gamma(i))$ for $0 \leq i \leq m$ (this is an enlargement of the collection of morphisms in $\mathcal{I}$, where we would require the stronger condition $\beta = \beta' \circ \gamma$).

Any morphism γ in $\mathcal{I}_+$ determines a map $Z_{\beta'} \to Z_\beta$, which carries a map of $[m']$-gapped objects $X'_{\beta'} \to X$ to the map of $[m]$-gapped objects $X'_\beta \to Y$ where

$$Y(i,j) = X(\gamma(i), \gamma(j)) \amalg_{X'(\beta'(\gamma(i)),\beta'(\gamma(j)))} X'(\beta(i), \beta(j)).$$

We therefore obtain a commutative diagram

$$\begin{array}{ccc} \varinjlim_{\beta \in \mathcal{I}_0^{\mathrm{op}}} Z_\beta & \longrightarrow & \varinjlim_{\beta \in \mathcal{I}^{\mathrm{op}}} Z_\beta \\ & \searrow \quad \swarrow & \\ & \varinjlim_{\beta \in \mathcal{I}_+^{\mathrm{op}}} Z_\beta. & \end{array}$$

To prove that the upper horizontal map is a homotopy equivalence, it will suffice to show that the lower horizontal maps are homotopy equivalences. This follows from the following combinatorial observation:

Lemma 9. *The inclusion maps $\mathcal{I}_0 \hookrightarrow \mathcal{I}_+$ and $\mathcal{I} \hookrightarrow \mathcal{I}_+$ are right cofinal.*

Proof. The right cofinality of $\mathcal{I}_0 \hookrightarrow \mathcal{I}_+$ follows from the fact that it admits a right adjoint (which carries an arbitrary map $\beta : [m] \to [n]$ to the constant map $[m] \to \{n\}$). To prove the right cofinality of the inclusion $\mathcal{I} \hookrightarrow \mathcal{I}_+$, we must work a little bit harder. Fix an object of $\mathcal{I}_+$ given by a map $\beta : [m] \to [n]$. Unwinding the definitions, we see that the overcategory $\mathcal{J} = \mathcal{I} \times_{\mathcal{I}_+} (\mathcal{I}_+)_{/\beta}$ can be identified with the category whose objects are nonempty finite linearly ordered sets $[k]$ equipped with a monotone map $\gamma : [k] \to P$, where $P = \{(i,j) \in [m] \times [n] : \beta(i) \leq j\}$. This category contains as a deformation retract the full subcategory spanned by the injective maps, whose geometric realization is homeomorphic to $|\mathrm{N}(P)|$. It will therefore suffice to show that the partially ordered set P is weakly contractible, which is clear because P has a smallest element. □

References

[1] Waldhausen, F. *Algebraic K-theory of spaces.*