Higher K-Theory of ∞-Categories (Lecture 16)

October 7, 2014

Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits; we will denote the coproduct on $\mathcal{C}$ by $(C, D) \mapsto C \vee D$ (to emphasize the analogy with pointed spaces). Recall that we defined the K-group $K_0(\mathcal{C})$ to be the free abelian group on symbols $[X]$ for $X \in \mathcal{C}$, modulo the relations $[X] = [X'] + [X'']$ for every cofiber sequence

$$X' \to X \to X''$$

in $\mathcal{C}$. Note that it is not necessary to demand in advance that this group be abelian: for any pair of objects $X, Y \in \mathcal{C}$, we have cofiber sequences

$$X \to X \vee Y \to Y$$

$$Y \to X \vee Y \to X$$

which give the relation $[X] + [Y] = [X \vee Y] = [Y] + [X]$.

Our goal in this lecture is to discuss higher K-groups of an ∞-category $\mathcal{C}$, following ideas of Waldhausen. Our basic plan is to construct a space W whose fundamental group is $K_0(\mathcal{C})$; we will then define the higher K-groups of $\mathcal{C}$ to be the higher homotopy groups of W. Let's have a look at what such a space should look like:

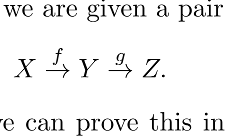
(a) The space W needs to have a base point (so that it makes sense to extract the homotopy groups of W).

(b) Every object $X \in \mathcal{C}$ needs to determine a path p_X in W which begins and ends in the base point, which we take be a representative of the homotopy class $[X] \in K_0(\mathcal{C}) \simeq \pi_1 W$.

(c) For every cofiber sequence

$$X' \to X \to X''$$

in the ∞-category $\mathcal{C}$, we will need a 2-simplex of W with boundary given by



which "witnesses" the relation $[X] = [X'] + [X'']$ in the fundamental group of W.

In what follows, it will be convenient to denote the cofiber of a map $f : X \to Y$ by Y/X; our basic relation in $K_0(\mathcal{C})$ can then be rewritten as $[Y] = [X] + [Y/X]$. From this, we can deduce analogous identities for "longer" filtrations. For example, suppose we are given a pair of morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

Then $[Z] = [X] + [Y/X] + [Z/Y]$. In fact, we can prove this in two different ways: first, we can use the cofiber sequences

$$X \to Z \to Z/X$$

$$Y/X \to Z/X \to Z/Y$$

to deduce $[Z] = [X] + [Z/X] = [X] + ([Y/X] + [Z/Y])$. Alternatively, we could use the cofiber sequences

$$Y \to Z \to Z/Y$$

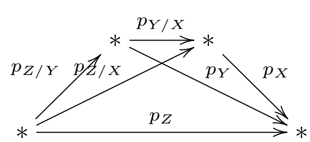
$$X \to Y \to Y/X$$

to deduce $[Z] = [Y] + [Z/Y] = ([X] + [Y/X]) + [Z/Y]$. If W is a space satisfying (a), (b), and (c), then these two proofs give two *a priori* different homotopies between the path p_Z and the concatenation of the paths p_X, $p_{Y/X}$, and $p_{Z/Y}$, which determines a map from a 2-sphere into W. Concretely, this two sphere is given by a map $\partial\,\Delta^3 \to W$, whose restriction to each fact of $\partial\,\Delta^3$ is the 2-simplex associated by assumption (c) to one of the cofiber sequences above. It is natural to demand the following "three-dimensional" analogue of (b) and (c):

(d) For every diagram

$$X \to Y \to Z$$

in $\mathcal{C}$, we should have a 3-simplex in W which we depict as



whose faces are the 2-simplices given by (c).

Moreover, we want to make an analogous "n-dimensional" demand for every n-step filtration

$$X_1 \to X_2 \to \cdots \to X_n$$

in the ∞-category $\mathcal{C}$. In order to say this properly, we need to get organized.

Definition 1. Let P be a partially ordered set, and let $P^{(2)} = \{(i,j) \in P \times P : i \leq j\}$. Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits. We define a *P-gapped object* of $\mathcal{C}$ to be a functor $X : \mathrm{N}(P^{(2)}) \to \mathcal{C}$ with the following properties:

(i) For every $i \in P$, the object $X(i,i)$ is a zero object of $\mathcal{C}$.

(ii) For every $i \leq j \leq k$ in $\mathcal{C}$, the diagram

$$\begin{array}{ccc} X(i,j) & \longrightarrow & X(i,k) \\ \downarrow & & \downarrow \\ X(j,j) & \longrightarrow & X(j,k) \end{array}$$

is a pushout square; by virtue of (i), this means that we have a cofiber sequence

$$X(i,j) \to X(i,k) \to X(j,k).$$

Note that if $f : P \to Q$ is a map of partially ordered sets, then f induces a map of simplicial sets $\mathrm{N}(P^{(2)}) \to \mathrm{N}(Q^{(2)})$. Composition with this map carries Q-gapped objects of $\mathcal{C}$ to P-gapped objects of $\mathcal{C}$.

For every integer $n \geq 0$, we let $[n]$ denote the linearly ordered set $\{0 < 1 < \ldots < n\}$. The collection of $[n]$-gapped objects of $\mathcal{C}$ form an ∞-category which we will denote by $\mathrm{Gap}_{[n]}(\mathcal{C})$. Let $S_n(\mathcal{C})$ denote the underlying Kan complex of $\mathrm{Gap}_{[n]}(\mathcal{C})$: that is, the simplicial set obtained from $\mathrm{Gap}_{[n]}(\mathcal{C})$ by removing all those edges which correspond to noninvertible morphisms in $\mathrm{Gap}_{[n]}(\mathcal{C})$ (along with all simplices which contain such edges). Any monotone map $[m] \to [n]$ induces a map of Kan complexes $S_n(\mathcal{C}) \to S_m(\mathcal{C})$; we may therefore regard $S_\bullet(\mathcal{C})$ as a simplicial Kan complex. We will refer to $S_\bullet(\mathcal{C})$ as *the Waldhausen construction* on $\mathcal{C}$.

Remark 2. By definition, an object of $\mathrm{Gap}_{[n]}(\mathcal{C})$ is a diagram

$$\begin{array}{ccccccc}
X(0,0) & \longrightarrow & X(0,1) & \longrightarrow & \cdots & \longrightarrow & X(0,n) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & X(1,1) & \longrightarrow & \cdots & \longrightarrow & X(1,n) \\
 & & & & \downarrow & & \downarrow \\
 & & & & \cdots & \longrightarrow & \cdots \\
 & & & & & & \downarrow \\
 & & & & & & X(n,n)
\end{array}$$

where every square is a pushout and the objects along the diagonal are zero.

Example 3. When $n = 0$, $\mathrm{Gap}_{[n]}(\mathcal{C})$ is the full subcategory of $\mathcal{C}$ spanned by the zero objects. Since $\mathcal{C}$ is pointed, this is a contractible Kan complex.

Example 4. When $n = 1$, $\mathrm{Gap}_{[n]}(\mathcal{C})$ is the ∞-category whose objects are diagrams

$$\begin{array}{ccc}
* & \longrightarrow & X \\
 & & \downarrow \\
 & & *'
\end{array}$$

where $X \in \mathcal{C}$ is arbitrary and $*$ and $*'$ are zero objects of $\mathcal{C}$. This ∞-category is equivalent to $\mathcal{C}$.

Example 5. When $n = 2$, $\mathrm{Gap}_{[n]}(\mathcal{C})$ is the ∞-category of diagrams

$$\begin{array}{ccccc}
* & \longrightarrow & X' & \longrightarrow & X \\
 & & \downarrow & & \downarrow \\
 & & *' & \longrightarrow & X'' \\
 & & & & \downarrow \\
 & & & & *''
\end{array}$$

where $*$, $*'$, and $*''$ are zero objects and square is a pushout. We can think of the data of such a diagram as determined by a pair of maps

$$X' \xrightarrow{f} X \xrightarrow{g} X''$$

together with a nullhomotopy of $g \circ f$, which exhibits X'' as a cofiber of f. More informally: $\mathrm{Gap}_{[2]}(\mathcal{C})$ is the ∞-category whose objects are cofiber sequences in $\mathcal{C}$. This is equivalent to the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C}) = \mathcal{C}^{\Delta^1}$ of morphisms in $\mathcal{C}$, since a cofiber sequence is determined up to equivalence by the morphism $f : X' \to X$.

Remark 6. More generally, for any $n \geq 0$, an $[n]$-gapped object

$$\begin{array}{ccccccc}
X(0,0) & \longrightarrow & X(0,1) & \longrightarrow & \cdots & \longrightarrow & X(0,n) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & X(1,1) & \longrightarrow & \cdots & \longrightarrow & X(1,n) \\
 & & & & \downarrow & & \downarrow \\
 & & & & \cdots & \longrightarrow & \cdots \\
 & & & & & & \downarrow \\
 & & & & & & X(n,n)
\end{array}$$

is determined by the sequence of maps

$$X(0,1) \to X(0,2) \to \cdots \to X(0,n):$$

the rest of the diagram can be functorially recovered by forming cofibers (note that $X(i,j)$ is the cofiber of the map $X(0,i) \to X(0,j)$). More precisely, evaluation on the pairs $(0,i)$ for $i > 0$ induces an equivalence of ∞-categories

$$\mathrm{Gap}_{[n]}(\mathcal{C}) \to \mathrm{Fun}(\Delta^{n-1}, \mathcal{C}).$$

Remark 7. Using Remark 6, we can describe the Waldhausen construction more informally as follows: for each $n \geq 0$, the Kan complex $S_n(\mathcal{C})$ is a classifying space for diagrams

$$X_1 \to X_2 \to \cdots \to X_n$$

in $\mathcal{C}$. However, the description of $S_\bullet(\mathcal{C})$ as a simplicial Kan complex is not completely apparent from this description. Most of the face maps $S_n(\mathcal{C}) \to S_{n-1}(\mathcal{C})$ correspond to simply "forgetting" one of the objects X_i in the diagram, but the 0th face map instead produces the diagram of cofibers

$$X_2/X_1 \to X_3/X_1 \to \cdots \to X_n/X_1$$

(which are only well-defined up to contractible ambiguity; this ambiguity is resolved in the definition of $\mathrm{Gap}_{[n]}(\mathcal{C})$ by specifying all of the relevant cofibers ahead of time).

Remark 8. From $S_\bullet(\mathcal{C})$ we can produce a new simplicial space $X_\bullet$, given by $X_n = S_{n+1}(\mathcal{C})$; passage from $S_\bullet(\mathcal{C})$ to $X_\bullet$ has the effect of "forgetting" the 0th face map in $S_\bullet(\mathcal{C})$. This simplicial space is equivalent to the one which assigns to each $[n]$ the underlying Kan complex of $\mathrm{Fun}(\Delta^n, \mathcal{C})$. The simplicial space $X_\bullet$ is an example of a *complete Segal space*. One can show that the homotopy theory of ∞-categories is equivalent to the homotopy theory of complete Segal spaces, with the equivalence implemented by the construction $\mathcal{C} \mapsto X_\bullet$. In particular, $\mathcal{C}$ is determined by $X_\bullet$ up to equivalence. Consequently, the Waldhausen construction $\mathcal{C} \mapsto S_\bullet(\mathcal{C})$ does not lose any information about the underlying ∞-category $\mathcal{C}$.

Definition 9 (Waldhausen K-theory of ∞-Categories)**.** Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits. We let $|S_\bullet(\mathcal{C})|$ denote the geometric realization of the Waldhausen construction on $\mathcal{C}$ (here we can work in the setting of topological spaces or simplicial sets; in the latter case, the geometric realization is given by the diagonal $[n] \mapsto S_n(\mathcal{C})_n$). The space $|S_\bullet(\mathcal{C})|$ has a canonical base point (up to contractible ambiguity), given by the map $S_0(\mathcal{C}) \to |S_\bullet(\mathcal{C})|$; we let $K(\mathcal{C})$ denote the loop space of $|S_\bullet(\mathcal{C})|$. For each integer $n \geq 0$, we let $K_n(\mathcal{C})$ denote the group $\pi_n K(\mathcal{C}) \simeq \pi_{n+1}|S_\bullet(\mathcal{C})|$.

Remark 10. Since the space $S_0(\mathcal{C})$ is contractible, the geometric realization $|S_\bullet(\mathcal{C})|$ is connected.

Remark 11. The space $|S_\bullet(\mathcal{C})|$ can be written as a direct limit of partial geometric realizations

$$\mathrm{sk}_0\,|S_\bullet(\mathcal{C})| \to \mathrm{sk}_1\,|S_\bullet(\mathcal{C})| \to \mathrm{sk}_2\,|S_\bullet(\mathcal{C})| \to \cdots$$

Here the 0-skeleton $\mathrm{sk}_0\,|S_\bullet(\mathcal{C})| = S_0(\mathcal{C})$ is contractible and the 1-skeleton $\mathrm{sk}_1\,|S_\bullet(\mathcal{C})|$ is the suspension of $S_1(\mathcal{C})$ (which is equivalent to the underlying Kan complex of $\mathcal{C}$). Since the inclusion $\mathrm{sk}_1\,|S_\bullet(\mathcal{C})| \to |S_\bullet(\mathcal{C})|$ is 1-connected, we have a surjection of fundamental groups

$$\pi_1 \Sigma S_1(\mathcal{C}) \to \pi_1|S_\bullet(\mathcal{C})|,$$

where the left hand side is the free group generated by one symbol $[X]$ for each connected component of $S_1(\mathcal{C})$ (which we can identify with an equivalence class of objects of $\mathcal{C}$) modulo the single relation $[*] = 1$. The inclusion $\mathrm{sk}_2\,|S_\bullet(\mathcal{C})| \to |S_\bullet(\mathcal{C})|$ is 2-connected, so all the relations in $\pi_1|S_\bullet(\mathcal{C})|$ come from connected components of $S_2(\mathcal{C})$ (which we can identify with cofiber sequences in $\mathcal{C}$). These relations say exactly that $[X] = [X'] + [X'']$ whenever we have a cofiber sequence

$$X' \to X \to X''$$

in $\mathcal{C}$. Consequently, the definition of $K_0(\mathcal{C})$ given in Definition 9 agrees with the definition given in Lecture 14.

It follows from the above discussion that the group $\pi_1|S_\bullet(\mathcal{C})| \simeq K_0(\mathcal{C})$ is abelian. In fact, this is no accident: the space $K(\mathcal{C})$ is an infinite loop space.

Remark 12. Let $\mathcal{C}$ and $\mathcal{D}$ be pointed ∞-categories which admit finite colimits and let $f : \mathcal{C} \to \mathcal{D}$ be a functor which preserves finite colimits. Then f induces a map of simplicial spaces $S_\bullet(\mathcal{C}) \to S_\bullet(\mathcal{D})$, hence also a map of K-theory spaces $K(\mathcal{C}) \to K(\mathcal{D})$.

If $\mathcal{C}$ and $\mathcal{D}$ are pointed ∞-categories which admit finite colimits, then the projection functors

$$\mathcal{C} \leftarrow \mathcal{C} \times \mathcal{D} \to \mathcal{D}$$

preserve finite colimits and therefore induce maps

$$K(\mathcal{C}) \leftarrow K(\mathcal{C} \times \mathcal{D}) \to K(\mathcal{D}).$$

These maps induce a homotopy equivalence (even an isomorphism, if we're sufficiently careful with our definitions) $K(\mathcal{C} \times \mathcal{D}) \simeq K(\mathcal{C}) \times K(\mathcal{D})$.

Note that the coproduct functor $\vee : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves finite colimits, and therefore induces a map

$$m : K(\mathcal{C}) \times K(\mathcal{C}) \simeq K(\mathcal{C} \times \mathcal{C}) \to K(\mathcal{C}).$$

Since the coproduct on $\mathcal{C}$ is coherently commutative and associative, the multiplication m is also coherently commutative and associative: that is, it is part of an E_∞-structure on the space $K(\mathcal{C})$.

The E_∞-structure on $K(\mathcal{C})$ determines a commutative monoid structure on the set $K_0(\mathcal{C}) = \pi_0 K(\mathcal{C})$. This coincides with the abelian group structure considered earlier (since $[X \vee Y] = [X] + [Y]$ in $K_0(\mathcal{C})$). It follows that $K(\mathcal{C})$ is a *grouplike* E_∞-space: that is, it is the 0th space of a connective spectrum. We will generally abuse notation and denote this spectrum also by $K(\mathcal{C})$.

Remark 13. One can obtain an explicit delooping of the space $K(\mathcal{C})$ by *iterating* the Waldhausen construction; we refer the reader to [2] for details.

References

[1] Barwick, C. *On the algebraic K-theory of higher categories.*

[2] Waldhausen, F. *Algebraic K-theory of spaces.*