Some Loose Ends (Lecture 12)

September 29, 2014

Earlier in this course, we stated the following result:

Theorem 1. *Let X and Y be finite polyhedra and let $f : X \to Y$ be a homotopy equivalence. The following conditions are equivalent:*

(1) *The map f is a simple homotopy equivalence.*

(2) *There exists a concordance $q : E \to [0, 1]$ from X to Y such that the induced homotopy equivalence $X \simeq q^{-1}\{0\} \to q^{-1}\{1\} \simeq Y$ is homotopic to f.*

The implication (2) $\Rightarrow$ (1) was proved in Lecture 6. We are now *almost* in a position to prove the converse. Let us begin by introducing a bit of (nonstandard) terminology.

Definition 2. Let X be a polyhedron and let D^n be the standard n-disk (regarded as a finite polyhedron). Given a PL embedding $e : D^n \hookrightarrow X$, the mapping cylinder $M(e) = X \amalg_{D^n} (D^n \times [0, 1])$ has the structure of a polyhedron and is equipped with a cell-like PL map $\pi : M(e) \to X$.

We will say that a PL map of finite polyhedra $Y \to X$ is a *polyhedral elementary contraction* if it has the form $\pi : M(e) \to X$ for some PL embedding $e : D^n \hookrightarrow X$, and a *polyhedral elementary expansion* if it is isomorphic to the canonical inclusion $X \hookrightarrow M(e)$ for some PL embedding $e : D^n \hookrightarrow X$. We say that f is a *polyhedral simple homotopy equivalence* if it is homotopic to a composition of polyhedral elementary expansions and polyhedral elementary contractions.

Claim 3. *Let $f : X \to Y$ be a map of finite polyhedra. Then f is a simple homotopy equivalence if and only if it is a polyhedral simple homotopy equivalence.*

To prove Claim 3, it suffices to show that f is a polyhedral simple homotopy equivalence if and only if it has vanishing Whitehead torsion $\tau(f) \in \mathrm{Wh}(\pi_1 X)$ (in the case where X is connected). This can be established by carrying out the argument sketched in Lecture 4 entirely in the setting of polyhedra.

Assuming Claim 3, we can prove the implication (1) $\Rightarrow$ (2) of Theorem 1. For this, it suffices to show that any polyhedral elementary contraction $f : X \to Y$ is a homotopy equivalence which arises from a concordance between X and Y. This is a special case of the following:

Proposition 4. *Let X and Y be finite polyhedra and let $f : X \to Y$ be a cell-like PL map. Then there exists a concordance $q : E \to [0, 1]$ from X to Y such that the induced homotopy equivalence $X \simeq q^{-1}\{0\} \to q^{-1}\{1\} \simeq Y$ is homotopic to f.*

Proof. Choose triangulations of X and Y that are compatible with f, so that f induces a Cartesian fibration $q : \Sigma(X) \to \Sigma(Y)$. Since f is cell-like, the fibers of q are weakly contractible. We complete the proof by setting $E = \mathrm{N}(\Sigma(Y) \amalg_q \Sigma(X))$ (which we proved to be a concordance in Lecture 10). □

Remark 5. In the second part of this course, we will give an alternative approach Theorem 1, which does not require revisiting Lecture 4 in the polyhedral setting.

In Lectures 10 and 11, we constructed a weak homotopy equivalence $\rho : \mathrm{N}(\mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{ns}})^{\mathrm{op}} \to \mathcal{M}$, where $\mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{ns}}$ is the category whose objects are finite nonsingular simplicial sets and whose morphisms are cell-like maps. We next show that the condition of nonsingularity is not essential.

Definition 6. Let $f : X \to Y$ be a map of finite simplicial sets (not necessarily nonsingular). We will say that f is *cell-like* if the induced map of topological spaces $|X| \to |Y|$ is cell-like (which is equivalent to the assertion that the fibers of $|f|$ are contractible).

We let $\mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{cl}}$ denote the category whose objects are finite simplicial sets and whose morphisms are cell-like maps.

Remark 7. Suppose we are given a pullback diagram of finite simplicial sets

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y. \end{array}$$

Then the associated diagram of geometric realizations is also a pullback diagram. It follows that f' is cell-like if f is; the converse holds if g is surjective.

Definition 8. Let X be a finite simplicial set. A *desingularization* of X is a finite nonsingular simplicial set Y equipped with a cell-like map $\pi : Y \to X$. We let $\mathcal{D}(X)$ denote the category whose objects are desingularizations $\pi : Y \to X$, and whose morphisms are commutative diagrams of cell-like maps

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi'} & \\ & X. & \end{array}$$

We will prove the following:

Proposition 9. *For every finite simplicial set X, the category $\mathcal{D}(X)$ is weakly contractible.*

Corollary 10. *The inclusion functor $\mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{ns}} \hookrightarrow \mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{cl}}$ induces a homotopy equivalence $|\,\mathrm{N}(\mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{ns}})| \hookrightarrow |\,\mathrm{N}(\mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{cl}})|$.*

Proof. Combine Proposition 8 with Quillen's Theorem A. □

Proposition 8 is an easy consequence of the following special case:

Lemma 11. *Let X be a finite simplicial set. Then there exists a desingularization of X.*

Proof of Proposition 8. Fix a desingularization $\pi : Y_0 \to X$. We define a functor $T : \mathcal{D}(X) \to \mathcal{D}(X)$ by the formula $T(Y) = Y \times_X Y_0$; note that $T(Y)$ is nonsingular since it is contained in the nonsingular simplicial set $Y \times Y_0$, and Remark 6 implies that $T(Y)$ is again a desingularization of X. The evident maps $Y \leftarrow T(Y) \to Y_0$ show that the identity map $\mathrm{id} : |\,\mathrm{N}(\mathcal{D})| \to |\,\mathrm{N}(\mathcal{D})|$ is homotopic to the constant map taking the value Y_0. □

Proof of Lemma 10. We proceed by induction on the number of nondegenerate simplices of X. If X is empty, there is nothing to prove; otherwise, we can write X as a pushout $X' \amalg_{\partial \Delta^n} \Delta^n$. The inductive hypothesis the implies that there exists a desingularization $Y' \to X'$. Set $K = Y' \times_{X'} \partial \Delta^n$; note that K is nonsingular since it is contained in the product $Y' \times \partial \Delta^n$. Choose an embedding $K \hookrightarrow C(K)$, where $C(K)$ is nonsingular and weakly contractible (for example, we can take $C(K)$ to be the join $K \star \Delta^0$). The diagonal map $K \to \Delta^n \times C(K)$ is then an embedding so that the mapping cylinder

$$M = K \times \Delta^1 \amalg_{K \times \{1\}} \Delta^n \times C(K)$$

is nonsingular. We have a commutative diagram of simplicial sets

$$\begin{array}{ccccc} Y' \times C(K) & \longleftarrow & K & \longrightarrow & M \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ X' & \longleftarrow & \partial\,\Delta^n & \longrightarrow & \Delta^n. \end{array}$$

Note that the upper horizontal maps are injective so that the pushout $Y = (Y' \times C(K)) \amalg_K M$ is a nonsingular simplicial set. We claim that the canonical map $\pi : Y \to X$ is cell-like. To prove this, choose any point $x \in |X|$; we claim that the inverse image $\pi^{-1}\{x\} \subseteq |Y|$ is contractible. If x does not belong to $|X'|$, then we have $\pi^{-1}\{x\} \simeq \gamma^{-1}\{x\}$, which is contractible, since γ is the composition of cell-like maps

$$M \to \Delta^n \times C(K) \to \Delta^n.$$

If x belongs to $|X'|$, let $Z \subseteq |\,\partial\,\Delta^n|$ be the inverse image of x; then $\pi^{-1}\{x\}$ is given by the pushout

$$\alpha^{-1}\{x\} \amalg_{\beta^{-1}Z} \gamma^{-1}Z.$$

The map α is a composition of cell-like maps

$$Y' \times C(K) \to Y' \to X',$$

so that $\alpha^{-1}\{x\}$ is contractible. It will therefore suffice to show that the inclusion $\beta^{-1}Z \hookrightarrow \gamma^{-1}Z$ is a homotopy equivalence. This is clear, since $\gamma^{-1}Z$ is homeomorphic to the mapping cylinder of $\beta^{-1}Z \to Z \times C(K)$ which is a homotopy equivalence since β is cell-like. □

Since $\mathcal{M}$ is a Kan complex, it follows from Corollary 9 that the map $\rho : \mathrm{N}(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}})^{\mathrm{op}} \to \mathcal{M}$ admits an extension $\overline{\rho} : \mathrm{N}(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{cl}})^{\mathrm{op}} \to \mathcal{M}$. It is not obvious how to construct such an extension explicitly because there there is no functorial way to endow the geometric realization $|X|$ with the structure of a polyhedron for an arbitrary finite simplicial set X. However, if we do not insist on working with polyhedra, then this problem goes away:

Definition 12. Let $Q = \prod_{n \geq 0} [0,1]$ be the Hilbert cube (or any other "sufficiently large" contractible space). We define a simplicial set $\mathcal{M}^+$ as follows: the n-simplices of $\mathcal{M}^+$ are subsets $E \subseteq \Delta^n \times Q$ with the following properties:

- As a topological space, E is a compact absolute neighborhood retract.
- The projection $E \to \Delta^n$ is a fibration.

Any choice of embedding $\mathbb{R}^\infty \hookrightarrow Q$ determines a map of simplicial sets $\mathcal{M} \hookrightarrow \mathcal{M}^+$; roughly speaking, this map is given by "forgetting" PL structures.

Example 13. Suppose we are given maps of topological spaces

$$X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n.$$

The *mapping simplex* of the sequence $\{X_i\}$ is defined to be the iterated pushout

$$X_0 \amalg_{X_1} (X_1 \times \Delta^1) \amalg_{X_2 \times \Delta^1} (X_2 \times \Delta^2) \amalg \cdots \amalg (X_n \times \Delta^n).$$

We will denote this mapping simplex by $M(X_0 \leftarrow \cdots \leftarrow X_n)$. There is an evident map

$$\pi : M(X_0 \leftarrow \cdots \leftarrow X_n) \to \Delta^n.$$

If each X_i is a compact ANR, one can show that $M(X_0 \leftarrow \cdots \leftarrow X_n)$ is a compact ANR. If, in addition, each of the maps $X_i \to X_{i-1}$ is cell-like, then the map π is a fibration: this follows from the main result of Lecture 8.

Construction 14. Suppose we are given an n-simplex σ of $\mathrm{N}(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{cl}})$, given by a sequence of cell-like maps

$$X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n.$$

Choosing an embedding of each $|X_i|$ into Q, we can regard the mapping simplex $M(|X_0| \leftarrow \cdots \leftarrow |X_n|)$ as a subset of $\Delta^n \times Q$ which defines an n-simplex of $\mathcal{M}^+$. This determines for us a map of simplicial sets

$$\rho' : \mathrm{N}(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{cl}})^{\mathrm{op}} \to \mathcal{M}^+ .$$

The maps ρ and ρ' are closely related:

Proposition 15. *The diagram of simplicial sets*

$$\begin{array}{ccc} \mathrm{N}(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}})^{\mathrm{op}} & \xrightarrow{\rho} & \mathcal{M} \\ \downarrow & & \downarrow \\ \mathrm{N}(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{cl}})^{\mathrm{op}} & \xrightarrow{\rho'} & \mathcal{M}^+ \end{array}$$

commutes up to homotopy.

Remark 16. In the final portion of this course, if we get there, we will show that the inclusion $\mathcal{M} \hookrightarrow \mathcal{M}^+$ is a homotopy equivalence (so that the diagram of Proposition 14 consists of homotopy equivalences).

Let us sketch the proof of Proposition 14. Suppose we are given an n-simplex σ of $\mathrm{N}(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}})^{\mathrm{op}}$, given by a sequence of cell-like maps

$$X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n$$

of *nonsingular* simplicial sets. We will abuse notation by identifying σ with its image in $\mathrm{N}(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{cl}})^{\mathrm{op}}$ and $\rho(\sigma)$ with its image in $\mathcal{M}^+$. We wish to relate $\rho(\sigma)$ to $\rho'(\sigma)$. Both can be identified with spaces which are fibered over the topological n-simplex Δ^n, given by the geometric realizations of certain finite simplicial sets: in the former case, we use the simplicial set

$$Y = \mathrm{N}(\Sigma(X_0) \amalg \Sigma(X_1) \amalg \cdots \amalg \Sigma(X_n)),$$

and in the latter case we use the mapping simplex

$$Z = M(X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_n)$$

(given by carrying out the construction of Example 12 in the category of simplicial sets rather than the category of topological spaces). Let W denote the simplicial set

$$M(\mathrm{Sd}(X_0) \leftarrow \mathrm{Sd}(X_1) \leftarrow \mathrm{Sd}(X_2) \leftarrow \cdots \leftarrow \mathrm{Sd}(X_n)).$$

Amalgamating the maps $\mathrm{Sd}(X_i) \to X_i$, we obtain a map $W \to Z$. There is also a canonical map $W \to Y$, given by amalgamating maps

$$\mathrm{Sd}(X_i) \times \Delta^i = \mathrm{N}(\Sigma(X_i) \times \{0, \ldots i\}) \xrightarrow{\theta_i} \mathrm{N}(\Sigma(X_0) \amalg \cdots \Sigma(X_n)),$$

where θ_i is given by natural map of posets carrying $\Sigma(X_i) \times \{j\}$ to $\Sigma(X_j)$ for $0 \leq j \leq i$.

Claim 17. *In the situation above, the maps $Y \leftarrow W \to Z$ are cell-like.*

Proof. The fact that the map $W \to Z$ is cell-like follows from the fact that each of the maps $\mathrm{Sd}(X_i) \to X_i$ is cell-like. To prove that the map $W \to Y$ is cell-like, we must show that for each point $y \in |Y|$ the fiber $F = |W| \times_{|Y|} \{y\}$ is cell-like. Without loss of generality we may assume that the image of y in Δ^n does not belong to Δ^{n-1} (otherwise, we can simply truncate our sequence of simplicial sets $\{X_i\}$). In this case, the space F is also the fiber of the map

$$\theta_n : \mathrm{N}(\Sigma(X_n) \times [n]) \to \mathrm{N}(\Sigma(X_0) \amalg \cdots \amalg \Sigma(X_n)).$$

Since the underlying map of posets is a Cartesian fibration, it suffices to check that the fibers of θ_n are weakly contractible: this follows from the fact that each of the maps $\Sigma(X_n) \to \Sigma(X_i)$ has weakly contractible fibers. $\square$

Consider now the 2-sided mapping cylinder

$$Y \amalg_{W \times \{0\}} (W \times \Delta^1) \amalg_{W \times \{1\}} Z.$$

This is a simplicial set whose geometric realization is equipped with a canonical map to $\Delta^n \times \Delta^1$; using Claim 16 and the results of Lecture 8, one can show that this map is a fibration. Choosing an embedding into the big contractible space Q, we obtain a map $\Delta^n \times \Delta^1 \to \mathcal{M}^+$ which is a homotopy from $\rho(\sigma)$ to $\rho'(\sigma)$. It is easy to see that these homotopies can be chosen to be compatible as σ varies and therefore supply a proof of Proposition 14.