Combinatorial Models for $\mathcal{M}$ (Lecture 10)

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Let $f : X \to Y$ be a map of finite nonsingular simplicial sets. In the previous lecture, we showed that the induced map $|f| : |X| \to |Y|$ is a fibration if and only if it satisfies the following combinatorial path lifting condition: for every simplex $\sigma_0 \in \Sigma(X)$ with image $\tau_0 = f(\sigma_0)$ in $\Sigma(Y)$ and every simplex $\tau \in \Sigma(Y)$ containing τ_0, the partially ordered set $\{\sigma \in \Sigma(X) : f(\sigma) = \tau \text{ and } \sigma_0 = \sigma \cap f^{-1}\tau_0\}$ is weakly contractible. Our first objective in this lecture is to apply this criterion in the special case where f arises from a map of partially ordered sets.

Proposition 1. *Let $f : P \to Q$ be a map of finite partially ordered sets. Assume that f is a Cartesian fibration. The following conditions are equivalent:*

(1) *The induced map $|\operatorname{N}(P)| \to |\operatorname{N}(Q)|$ is a fibration.*

(2) *For every $q' \leq q$ in Q, the induced map of fibers $P_q \to P_{q'}$ is left cofinal.*

Proof. Let $\operatorname{Chain}(P) = \Sigma(\operatorname{N}(P))$ denote the partially ordered set of nonempty chains in P, and similarly for Q. Fix a simplex $\sigma_0 \in \operatorname{Chain}(P)$ having image $\tau_0 \in \operatorname{Chain}(Q)$. For each chain τ in $\operatorname{Chain}(Q)$ containing τ_0, let

$$S_\tau = \{\sigma \in \operatorname{Chain}(P) | f(\sigma) = \tau \text{ and } \sigma \cap f^{-1}\tau_0 = \sigma_0\}.$$

The criterion of the previous lecture shows that (1) holds if and only if each of the partially ordered sets S_τ is weakly contractible.

Assume first that (1) is satisfied. Let $q' < q$ in Q; we wish to show that the map $P_q \to P_{q'}$ is left cofinal. For this, pick $p' \in P_{q'}$; we need to show that $T = \{p \in P_q : p' \leq p\}$ is weakly contractible. Taking $\sigma_0 = \{p'\}$, $\tau_0 = \{q'\}$, and $\tau = \{q', q\}$, we see that the set S_τ above can be identified with $\operatorname{Chain}(T)$. Condition (1) implies that S_τ is weakly contractible, so that T is likewise weakly contractible (since $\operatorname{N}(S_\tau) \simeq \operatorname{Sd}\operatorname{N}(T)$).

We now prove that (2) $\Rightarrow$ (1). Choose $\sigma_0 \in \operatorname{Chain}(P)$ and $\tau \in \operatorname{Chain}(Q)$ as above; we wish to show that S_τ is weakly contractible. If $\tau = \tau_0$, then there is nothing to prove. Otherwise, choose an element q which belongs to $\tau - \tau_0$ and set $\tau' = \tau - \{q\}$. Note that if we are given a simplex $\tau' \in \operatorname{Chain}(Q)$ with $\tau_0 \subseteq \tau' \subseteq \tau$, then the construction $\sigma \mapsto \sigma \cap f^{-1}\tau'$ determines a Cartesian fibration $S_\tau \to S_{\tau'}$. Proceeding inductively, we may assume that $S_{\tau'}$ is weakly contractible; it then suffices to show that the fibers of the map $S_\tau \to S_{\tau'}$ are weakly contractible. Replacing τ_0 by τ' we may reduce to the case where τ is obtained from τ_0 by adding a single element q.

Then τ corresponds to a chain

$$\{q_{-m} < \ldots < q_{-1} < q < q_1 < \cdots < q_n\}$$

where either m or n could be zero (but not both), and σ_0 corresponds to a chain $\{p_0 < \ldots < p_k\}$ lying over $\{q_{-m} < \ldots < q_{-1} < q_1 < \cdots < q_n\}$. Let p_- be the largest element of σ_0 which lies over q_{-1} (if $m \neq 0$) and let p_+ be the largest element of σ_0 which lies over q_1 (if $n \neq 0$). Unwinding the definitions, we see that $\operatorname{N}(S_\tau)$ can be identified with the subdivision of the nerve of the poset

$$\begin{cases} \{p \in P_q : p \leq p_+\} \text{ if } m = 0 \\ \{p \in P_q : p_- \leq p\} \text{ if } n = 0 \\ \{p \in P_q : p_- \leq p \leq p_+\} \text{ if } m, n \neq 0. \end{cases}$$

Since f is a Cartesian fibration, the partially ordered sets of the first and third type have largest elements. It will therefore suffice to consider the case where $n = 0$. Let $\alpha : P_q \to P_{q_{-1}}$ be the map induced by the inequality $q_{-1} < q$. Then the poset in question can be identified with $\{p \in P_q : \alpha(p) \geq p_-\}$, which is weakly contractible by virtue of our assumption that α is left cofinal. $\square$

Remark 2. It follows from Proposition 1 that if $f : X \to Y$ is a cell-like PL map of polyhedra, then X and Y are concordant (set $Q = \{0 < 1\}$ and apply Proposition 1 to the posets of simplices for compatible triangulations of X and Y). This *almost* proves that concordance of polyhedra is equivalent to simple homotopy equivalence (it would supply a complete proof if we had worked in the category of polyhedra at the outset, and considered only elementary expansions and elementary contractions in piecewise linear setting).

Let us now put Proposition 1 to work.

Definition 3. Let $\mathcal{C}_{\mathrm{cof}}$ denote the category whose objects are finite partially ordered sets and whose morphisms are left cofinal maps of partially ordered sets.

Consider the simplicial set $\mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})$. By definition, an n-simplex σ of $\mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})$ is a diagram of left cofinal maps

$$P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots \leftarrow P_n$$

of finite partially ordered sets. In this case, the disjoint union $P(\sigma) = \coprod_{0 \leq i \leq n} P_i$ can be regarded as a partially ordered set equipped with a Cartesian fibration $P(\sigma) \to [n]$ which satisfies the hypotheses of Proposition 1. It follows that the induced map $|\mathrm{N}(P(\sigma))| \to |\mathrm{N}([n])| \simeq \Delta^n$ is a piecewise linear fibration.

Let $\widetilde{\mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})}$ be the simplicial set whose n-simplices consist of n-simplices σ of $\mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})$ together with a PL embedding $|\mathrm{N}(P(\sigma))| \hookrightarrow \mathbb{R}^\infty$. There is an evident projection map $\phi : \widetilde{\mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})} \to \mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})$ given by forgetting the embeddings into $\mathbb{R}^\infty$, which is easily seen to be a trivial Kan fibration. We also have a map $\psi : \widetilde{\mathrm{N}(\mathcal{C}_{\mathrm{cof}}^{\mathrm{op}})}) \to \mathcal{M}$ which is given by forgetting the diagram

$$P_0 \leftarrow P_1 \leftarrow \cdots \leftarrow P_n$$

and remembering only the image of the map $|P(\sigma)| \to \Delta^n \times \mathbb{R}^\infty$. Composing ψ with a section of ϕ and using the canonical isomorphism $\mathcal{M} \simeq \mathcal{M}^{\mathrm{op}}$, we obtain a map of simplicial sets $\mathrm{N}(\mathcal{C}_{\mathrm{cof}}) \to \mathcal{M}$ which is well-defined up to homotopy.

Variant 4. Let $\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}}$ denote the category whose objects are finite nonsingular simplicial sets and whose morphisms are cell-like maps. The construction $X \mapsto \Sigma(X)$ determines a functor $\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}} \to \mathcal{C}_{\mathrm{cof}}$. Composing with the construction of Definition 3, we obtain a map of simplicial sets $\mathrm{N}(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}}) \to \mathcal{M}$.

Variant 5. Let $\mathcal{C}_{\mathrm{cell}}$ denote the subcategory of $\mathcal{C}_{\mathrm{cof}}$ whose objects are finite partially ordered sets and whose morphisms are Cartesian fibrations with weakly contractible fibers (note that any such map is left cofinal). Then the functor

$$\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}} \to \mathcal{C}_{\mathrm{cof}}$$
$$X \mapsto \Sigma(X)$$

factors through $\mathcal{C}_{\mathrm{cell}}$.

We can now state the first main theorem of this course:

Theorem 6. *The maps of simplicial sets*

$$\mathrm{N}(\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ns}}) \xrightarrow{\alpha} \mathrm{N}(\mathcal{C}_{\mathrm{cell}}) \xrightarrow{\beta} \mathrm{N}(\mathcal{C}_{\mathrm{cof}}) \xrightarrow{\gamma} \mathcal{M}$$

are all weak homotopy equivalences.

Theorem 6 supplies several different "purely combinatorial" definitions of simple homotopy theory.

We begin by discussing the easy parts of Theorem 6.

Proposition 7. *The map* $\alpha : \mathrm{N}(\mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{ns}}) \to \mathrm{N}(\mathcal{C}_{\mathrm{cell}})$ *is a weak homotopy equivalence.*

Proof. In the previous lecture, we saw that a Cartesian fibration $P \to Q$ with weakly contractible fibers induces a cell-lie map $\mathrm{N}(P) \to \mathrm{N}(Q)$. Consequently, the construction $P \mapsto \mathrm{N}(P)$ can be regarded as a functor from $\mathcal{C}_{\mathrm{cell}}$ to $\mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{ns}}$. This functor defines a map of simplicial sets $\alpha' : \mathrm{N}(\mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{ns}}) \to \mathrm{N}(\mathcal{C}_{\mathrm{cell}})$. We will show that α' is homotopy inverse to α.

Consider first the composition $\alpha' \circ \alpha$, which is given by taking the nerve of the subdivision functor $\mathrm{Sd} : \mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{ns}} \to \mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{ns}}$. For any nonsingular simplicial set X, there is a canonical map $\mathrm{Sd}(X) \to X$, which carries a nondegenerate n-simplex of $\mathrm{Sd}(X)$ given by a chain

$$\sigma_0 \subseteq \cdots \subseteq \sigma_n$$

to the n-simplex of X given by the composite map

$$\Delta^n \xrightarrow{f} \sigma_n \to X,$$

where f carries the ith vertex of Δ^n to the last vertex of σ_i. We claim that this map is cell-like (so that it determines a homotopy from $\alpha' \circ \alpha$ to the identity). Invoking the criterion of the previous lecture, we are reduced to showing that the map of posets $\Sigma(\mathrm{Sd}(X)) \to \Sigma(X)$ has weakly contractible fibers. Unwinding the definitions, we see that the inverse image of a simplex $\sigma \in \Sigma(X)$ can be identified with the partially ordered set S of chains

$$\vec{\tau} = \{\tau_0 \subset \sigma_1 \subset \cdots \subset \tau_n\}$$

in $\Sigma(X)$ which have the property that $\sigma \subseteq \tau_n$ and the vertices of σ are precisely those vertices of τ_n which occur as the final vertex of some τ_i.

Let d be the dimension of σ and for $0 \leq i \leq d$ let $\sigma_i \subseteq \sigma$ be the facet spanned by the first $(i+1)$-vertices of σ. Let $S_i \subseteq S$ be the subset consisting of those chains $\vec{\tau}$ which satisfy $\tau_j = \sigma_j$ for $j \leq i$, and let $S'_i \subseteq S_i$ be the further subset consisting of those chains $\vec{\tau}$ where $\sigma_{i+1} \subseteq \tau_{i+1}$; by convention, we let $S'_{-1} = S$. Note that each S_i is a deformation retract of S'_{i-1} (by the construction $\vec{\tau} \mapsto \vec{\tau} \cup \{\sigma_i\}$) and that each S'_i is a deformation retract of S_i (by the construction $\vec{\tau} \mapsto \{\tau_j : j \leq i \text{ or } \sigma_{i+1} \subseteq \tau_j\}$. It follows that S is weakly homotopy equivalent to S_d and therefore weakly contractible (since S_d has a smallest element given by the chain $\{\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_d\}$.

Let us now consider the other composition $\alpha \circ \alpha'$, which is induced by the functor $\mathcal{C}_{\mathrm{cell}} \to \mathcal{C}_{\mathrm{cell}}$ given by $P \mapsto \mathrm{Chain}(P)$. We claim that this induces a map $\mathrm{N}(\mathcal{C}_{\mathrm{cell}}) \to \mathrm{N}(\mathcal{C}_{\mathrm{cell}})$ which is homotopic to the identity. To see this, we assign to each finite partially ordered set P another finite partially ordered set $T(P) = \{(p, \sigma) \in P \times \mathrm{Chain}(P) : (\forall p' \in \sigma) p \leq p'\}$. It is not difficult to see that T determines a functor from $\mathcal{C}_{\mathrm{cell}}$ to itself and that the projection maps

$$P \leftarrow T(P) \to \mathrm{Chain}(P)$$

are Cartesian fibrations with weakly contractible fibers (those fibers are posets of the form $\{q \in P : q \leq p\}$ and $\mathrm{Chain}(\{q \in P : q \geq p\})$ for some $p \in P$, respectively). □

Proposition 8. *The map* $\beta : \mathrm{N}(\mathcal{C}_{\mathrm{cell}}) \to \mathrm{N}(\mathcal{C}_{\mathrm{cof}})$ *is a weak homotopy equivalence.*

The proof of Proposition 8 is a bit more involved. First, we recall that the subdivision $\mathrm{Sd}(X)$ can be defined for an arbitrary simplicial set by setting

$$\mathrm{Sd}(X) = \varinjlim_{\Delta^n \to X} \mathrm{Sd}(\Delta^n)$$

(this definition agrees with our earlier definition $\mathrm{Sd}(X) = \mathrm{N}(\Sigma(X))$ in the case where X is nonsingular). For any X, there is a canonical map $\rho_X : \mathrm{Sd}(X) \to X$ which is given by the colimit of the maps $\mathrm{Sd}(\Delta^n) \to \Delta^n$

which assign to each facet of Δ^n its final vertex. We saw in the proof of Proposition 7 that this map is a weak homotopy equivalence (in fact, even cell-like) when X is nonsingular. It follows formally (working simplex-by-simplex) that ρ_X is always a weak homotopy equivalence.

We now define a map $\delta : \mathrm{Sd}(\mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})) \to \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cell}})$. To give such a map, we must associate to every n-simplex $e : \Delta^n \to \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})$ a map $\mathrm{Sd}(\Delta^n) \to \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cell}})$, which we can identify with a functor $v : \Sigma(\Delta^n)^{\mathrm{op}} \to \mathcal{C}_{\mathrm{cell}}$. The simplex e is given by a diagram

$$P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots \leftarrow P_n$$

of finite partially ordered sets and left cofinal maps. Let $P = \bigcup P_i$ be equipped with the partial ordering described in Definition 3. Given a pair of chains $\sigma, \sigma' \in \mathrm{Chain}(P)$, we will write $\sigma \leq \sigma'$ if $p \leq p'$ for every $p \in \sigma$ and $p' \in \sigma'$ (of course, it suffices to check this when p is the largest element of σ and p' is the least element of σ').

Let τ be an m-dimensional facet of Δ^n, corresponding to a set

$$\{i_0 < \cdots < i_m\} \subseteq \{0 < 1 < \cdots < n\}.$$

We define $v : \Sigma(\Delta^n)^{\mathrm{op}} \to \mathcal{C}_{\mathrm{cell}}$ by the formula

$$v(\tau) = \{(\sigma_0, \ldots, \sigma_m) \in \mathrm{Chain}(P_{i_0}) \times \cdots \times \mathrm{Chain}(P_{i_m}) | \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_m.\}$$

It is easy to see that every inclusion of facets $\tau' \subseteq \tau$ induces a Cartesian fibration $v(\tau) \to v(\tau')$, and we saw in Proposition 1 that such maps have weakly contractible fibers. This completes the construction of the map δ.

We claim that δ supplies a homotopy inverse to β. More precisely, we claim that the composite maps

$$\mathrm{Sd}(\mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cell}})) \stackrel{\mathrm{Sd}(\beta)}{\to} \mathrm{Sd}(\mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})) \stackrel{\delta}{\to} \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cell}})$$

and

$$\mathrm{Sd}(\mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})) \stackrel{\delta}{\to} \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cell}}) \stackrel{\beta}{\to} \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})$$

are homotopic to the natural maps

$$\rho_{\mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cell}})} : \mathrm{Sd}(\mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cell}})) \to \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cell}})$$

$$\rho_{\mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})} : \mathrm{Sd}(\mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})) \to \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}}).$$

To prove the second of these statements, we will construct a homotopy

$$h : \mathrm{Sd}(\mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}}) \times \Delta^1) \to \mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}}).$$

To define h, we need to supply for each n-simplex $e : \Delta^n \to \mathrm{Sd}(\mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cell}})$ a functor $v : \Sigma(\Delta^n \times \Delta^1)^{\mathrm{op}} \to \mathcal{C}_{\mathrm{cell}}$. Let us regard e as given by a diagram

$$P_0 \leftarrow \cdots \leftarrow P_n$$

of left cofinal maps, and let $P = \bigcup P_i$ be defined as above. For $p \in P$ and $\sigma \in \mathrm{Chain}(p)$, we will write $p \leq \sigma$ if $p \leq q$ for each $q \in \sigma$. For $p \in P_i$ and $q \in P_j$, we will write $p \leftarrow q$ if $i \leq j$ and p is the image of q under the map $P_j \to P_i$ (note that this implies that $p \leq q$).

Let τ be a simplex of $\Delta^n \times \Delta^1$, which we can identify with a chain

$$\{(i_1, 0) < \cdots < (i_m, 0) < (j_1, 1) < \cdots < (j_{m'}, 1)\} \subseteq [n] \times [1].$$

We then define

$$v(\tau) = \{(p_1, \ldots, p_m, \sigma_1, \ldots, \sigma'_m) \in P_{i_1} \times \cdots \times P_{i_m} \times \mathrm{Chain}(P_{j_1}) \times \cdots \times \mathrm{Chain}(P_{j_{m'}}) | p_1 \leftarrow p_2 \leftarrow \cdots \leftarrow p_m \leq \sigma_1 \leq \cdots \leq \sigma_{m'}\}.$$

Every inclusion $\tau' \subseteq \tau$ induces a map of partially ordered sets $\theta : v(\tau) \to v(\tau')$. We claim that each of these maps is left cofinal. To prove this, we may assume without loss of generality that τ' is obtained from τ by omitting a single vertex. If this vertex has the form $(i_a, 0)$ for $a < m$, then θ is an isomorphism and there is nothing to prove. If the vertex has the form $(j_b, 1)$, then θ is a Cartesian fibration whose fibers are of the type analyzed in the proof of Proposition 1, and therefore weakly contractible. it therefore suffices to treat the case where τ' is obtained from τ by omitting the vertex $(i_m, 0)$ (in which case we must have $m > 0$). If $m = 1$, then θ is a Cartesian fibration whose fibers have the form $\{p \in P_{i_m} : p \leq p'\}$ for some $p' \in P_{i_m}$, and is therefore a weak homotopy equivalence. Let us therefore assume that $m > 1$. Fix an element of $v(\tau')$ given by a sequence $(p_1, \ldots, p_{m-1}, \sigma_1, \ldots, \sigma_m)$; we wish to show that the partially ordered set

$$S = \{(p'_1, \ldots, p'_m, \sigma'_1, \ldots, \sigma'_m) \in v(\tau) | p_{m-1} \leq p'_{m-1} \text{ and } \sigma_j \subseteq \sigma'_j\}$$

Let S' be the subset of S consisting of those tuples where $\sigma'_j = \sigma_j$ for all j. The inclusion $S' \hookrightarrow S$ admits a right adjoint and is therefore a weak homotopy equivalence. It will therefore suffice to show that S' is weakly contractible. Unwinding the definitions, we see that S' either has the form

$$\{p'_m \in P_{i_m} : p_{m-1} \leq p'_m \leq \sigma'_1\} \text{ or } \{p'_m \in P_{i_m} : p_{m-1} \leq p'_m\}$$

depending on whether or not $m' = 0$. In the former case, S' has a largest element; in the latter, it is weakly contractible by virtue of the left cofinality of $P_{i_m} \to P_{i_{m-1}}$.

The above analysis shows that the homotopy h is a well-defined map of simplicial sets. It follows from unwinding the definitions that the restriction of h to $\mathrm{Sd}(\mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}}) \times \{0\})$ is given by $\rho_{\mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}})}$ and the restriction of h to $\mathrm{Sd}(\mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cof}}) \times \{0\})$ agrees with $\beta \circ \delta$.

To construct the other homotopy, it suffices to observe that if the maps

$$P_0 \leftarrow P_1 \leftarrow \cdots \leftarrow P_n$$

are Cartesian fibrations with contractible fibers, then each of the maps $v(\tau) \to v(\tau')$ has the same property (arguing as above, this reduces easily to the case where τ' is obtained from τ by omitting the vertex $(i_m, 0)$ and where $m > 1$, in which case the desired result can be deduced from the assumption that the map $P_{i_m} \to P_{i_{m-1}}$ is a Cartesian fibration). Consequently, the restriction of h to $\mathrm{Sd}(\mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cell}}) \times \Delta^1)$ factors through $\mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cell}})$, and determines a homotopy from $\rho_{\mathrm{N}(\mathcal{C}^{\mathrm{op}}_{\mathrm{cell}})}$ to $\delta \circ \mathrm{Sd}(\beta)$.