Lecture 9: Sheaves

February 11, 2018

Recall that a category $\mathcal{X}$ is a *topos* if there exists an equivalence $\mathcal{X} \simeq \mathrm{Shv}(\mathcal{C})$, where $\mathcal{C}$ is a small category (which can be assumed to admit finite limits) equipped with a Grothendieck topology. In this lecture, we will describe some of the important categorical properties enjoyed by topoi. Our main goal is to prove the following:

Theorem 1. *Every topos is a pretopos.*

The main ingredient we will need in the proof of Theorem 1 is the following:

Theorem 2. *Let $\mathcal{C}$ be a small category which admits finite limits and is equipped with a Grothendieck topology. The inclusion functor $\mathrm{Shv}(\mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ admits a left adjoint $L : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Shv}(\mathcal{C})$. Moreover, the functor L preserves finite limits.*

For any presheaf $\mathscr{F}$, we will refer to $L\mathscr{F}$ as the *sheafification of $\mathscr{F}$*.

Let us first show that Theorem 2 implies Theorem 1. Let $\mathcal{X}$ be a topos, and write $\mathcal{X} = \mathrm{Shv}(\mathcal{C})$ where $\mathcal{C}$ is a small category (which admits finite limits). We proceed in several steps.

Lemma 3. *The category $\mathcal{X}$ admits arbitrary inverse limits.*

Proof. We first note that the category $\mathcal{S}\mathrm{et}$ admits arbitrary inverse limits. It follows that the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ of presheaves admits arbitrary inverse limits. These are just computed pointwise (by the formula $(\varprojlim \mathscr{F}_\alpha)(C) = \varprojlim (\mathscr{F}_\alpha(C))$ for $C \in \mathcal{C}$). It follows from this description that if each $\mathscr{F}_\alpha$ is a sheaf, then so is the inverse limit $\mathscr{F} = \varprojlim \mathscr{F}_\alpha$; in this case, $\mathscr{F}$ is also a limit of the diagram $\{\mathscr{F}_\alpha\}$ in the category $\mathrm{Shv}(\mathcal{C})$. □

Lemma 4. *The category $\mathcal{X}$ admits arbitrary colimits (that is, direct limits).*

Proof. Since the category $\mathcal{S}\mathrm{et}$ has all colimits, it follows that $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ admits arbitrary colimits (which are again computed pointwise by the formula $(\varinjlim \mathscr{F}_\alpha)(C) = \varinjlim (\mathscr{F}_\alpha(C))$ for $C \in \mathcal{C}$). This construction generally does not carry sheaves to sheaves. However, we can obtain a sheaf $\mathscr{F}$ by applying the sheafification functor L to $\varinjlim \mathscr{F}_\alpha$. Note that, for any sheaf $\mathscr{G}$, we have canonical bijections

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Shv}(\mathcal{C})}(\mathscr{F}, \mathscr{G}) &= \mathrm{Hom}_{\mathrm{Shv}(\mathcal{C})}(L \varinjlim \mathscr{F}_\alpha, \mathscr{G}) \\ &\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})}(\varinjlim \mathscr{F}_\alpha, \mathscr{G}) \\ &\simeq \varprojlim \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})}(\mathscr{F}_\alpha, \mathscr{G}). \end{aligned}$$

It follows that if each $\mathscr{F}_\alpha$ is a sheaf, then $L \varinjlim (\mathscr{F}_\alpha)$ is a colimit of the diagram $\{\mathscr{F}_\alpha\}$ in the category $\mathrm{Shv}(\mathcal{C})$. □

Lemma 5. *Every equivalence relation in $\mathcal{X}$ is effective.*

Proof. Let $\mathscr{F}$ be a sheaf and let $\mathscr{R} \subseteq \mathscr{F} \times \mathscr{F}$ be an equivalence relation on $\mathscr{F}$. Then, for every object $C \in \mathcal{C}$, we can view $\mathscr{R}(C)$ as an equivalence relation on $\mathscr{F}(C)$. Set $\mathscr{E}(C) = \mathscr{F}(C)/\mathscr{R}(C)$. The construction $C \mapsto \mathscr{E}(C)$ determines a presheaf of sets on $\mathcal{C}$, given by the coequalier of the two projection maps $\pi, \pi' : \mathscr{R} \to \mathscr{F}$ in the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$. It follows that the sheafification $L\mathscr{E}$ is a coequalizer of π and π' in the category $\mathrm{Shv}(\mathcal{C})$. To show that $\mathscr{R}$ is effective, we must show that the canonical map $\delta : \mathscr{R} \to \mathscr{F} \times_{L\mathscr{E}} \mathscr{F}$ is an isomorphism. Note that we can identify δ with the canonical map

$$L(\mathscr{F} \times_{\mathscr{E}} \mathscr{F}) \to L\mathscr{F} \times_{L\mathscr{E}} L\mathscr{F},$$

which is an isomorphism because the sheafification functor L preserves finite limits. □

Lemma 6. *Coproducts in $\mathcal{X}$ are disjoint.*

Proof. Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves, and let $\mathscr{F} \amalg \mathscr{G}$ denote the coproduct of $\mathscr{F}$ and $\mathscr{G}$ *in the category of presheaves* (given by the formula $(\mathscr{F} \amalg \mathscr{G})(C) = \mathscr{F}(C) \amalg \mathscr{G}(C)$). Then the sheafification $L(\mathscr{F} \amalg \mathscr{G})$ is a coproduct of $\mathscr{F}$ and $\mathscr{G}$ in the category of sheaves. We wish to show that the fiber product

$$\mathscr{F} \times_{L(\mathscr{F} \amalg \mathscr{G})} \mathscr{G}$$

is an initial object of $\mathrm{Shv}(\mathcal{C})$. Since the functor L preserves finite limits, we can identify this fiber product with $L(\mathscr{F} \times_{\mathscr{F} \amalg \mathscr{G}} \mathscr{G}) = L(\emptyset)$, where $\emptyset$ denotes the initial object of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$. We conclude by observing that L carries the initial presheaf to the initial sheaf. □

Lemma 7. *Colimits in the category $\mathcal{X}$ are universal. That is, for every morphism $f : \mathscr{F} \to \mathscr{G}$ in the category $\mathcal{X}$ and every diagram $\{\mathscr{G}_\alpha\}$ in $\mathcal{X}_{/\mathscr{G}}$, the canonical map*

$$\varinjlim_{\alpha} (\mathscr{F} \times_{\mathscr{G}} \mathscr{G}_\alpha) \to \mathscr{F} \times_{\mathscr{G}} (\varinjlim \mathscr{G}_\alpha)$$

is an isomorphism (where the colimits are taken in the category of sheaves*).*

Exercise 8. Check that Lemma 7 holds when $\mathcal{X} = \mathcal{S}\mathrm{et}$ is the category of sets.

Proof of Lemma 7. Let us break with the notation used in the statement of Lemma 7 and instead write $(\varinjlim \mathscr{G}_\alpha)$ and $\varinjlim_\alpha (\mathscr{F} \times_{\mathscr{G}} \mathscr{G}_\alpha)$ for the appropriate colimits formed in the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ of *presheaves*. In this case, everything is computed pointwise, so that Exercise 8 shows that the map

$$\varinjlim_{\alpha} (\mathscr{F} \times_{\mathscr{G}} \mathscr{G}_\alpha) \to \mathscr{F} \times_{\mathscr{G}} (\varinjlim \mathscr{G}_\alpha)$$

is an isomorphism of *presheaves*. Applying the sheafification functor L (and using that it commutes with finite limits), we deduce that the canonical map

$$\begin{aligned} L \varinjlim_{\alpha} (\mathscr{F} \times_{\mathscr{G}} \mathscr{G}_\alpha) &\to L(\mathscr{F} \times_{\mathscr{G}} (\varinjlim \mathscr{G}_\alpha)) \\ &\to L\mathscr{F} \times_{L\mathscr{G}} L(\varinjlim \mathscr{G}_\alpha) \\ &\simeq \mathscr{F} \times_{\mathscr{G}} L(\varinjlim \mathscr{G}_\alpha). \end{aligned}$$

is an isomorphism, which is the content of Lemma 7. □

Lemma 9. *Suppose we are given a pullback diagram*

$$\begin{array}{ccc} \mathscr{F}' & \longrightarrow & \mathscr{F} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathscr{G}' & \longrightarrow & \mathscr{G} \end{array}$$

in the category $\mathcal{X}$. If f is an effective epimorphism, then so is f'.

Proof. Set $\mathscr{R} = \mathscr{F} \times_{\mathscr{G}} \mathscr{F}$ and set $\mathscr{R}' = \mathscr{F}' \times_{\mathscr{G}'} \mathscr{F}'$. Our assumption that f is an effective epimorphism guarantees that we have a coequalizer diagram

$$\mathscr{R} \rightrightarrows \mathscr{F} \longrightarrow \mathscr{G}$$

Pulling back along the map $\mathscr{G}' \to \mathscr{G}$ and applying Lemma 7, we obtain a coequalizer diagram

$$\mathscr{R}' \rightrightarrows \mathscr{F}' \longrightarrow \mathscr{G}'$$

so that f' is also an effective epimorphism. □

Proof of Theorem 1. Let $\mathcal{X} \simeq \mathrm{Shv}(\mathcal{C})$ be a topos. Then $\mathcal{X}$ admits finite limits (Lemma 3), equivalence relations in $\mathcal{X}$ are effective (Lemma 5), $\mathcal{X}$ admits disjoint coproducts (Lemmas 4 and 6), pullbacks of effective epimorphisms are effective effective epimorphisms (Lemma 9), and coproducts are preserved by pullback (Lemma 7). In particular, $\mathcal{X}$ is a pretopos. □

Exercise 10. Let $\mathcal{C}$ be a pretopos and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory. Suppose that that inclusion functor $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ admits a left adjoint $L : \mathcal{C} \to \mathcal{C}_0$ which preserves finite limits. Show that $\mathcal{C}_0$ is also a pretopos (when applied to the inclusion $\mathrm{Shv}(\mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$, this shows that Theorem 2 implies Theorem 1).

We now turn to the proof of Theorem 2. For the remainder of this lecture, we fix a small category $\mathcal{C}$ with finite limits, which admits a Grothendieck topology. Recall that a functor $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ is a *sheaf* if, for every covering $\{U_i \to X\}$ in $\mathcal{C}$, the diagram

$$\mathscr{F}(X) \longrightarrow \prod_{i \in I} \mathscr{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathscr{F}(U_i \times_X U_j)$$

is an equalizer in the category of sets. In what follows, we will refer to an element $s \in \mathscr{F}(Y)$ as a *section of $\mathscr{F}$ over Y*; given a morphism $Y' \to Y$ in $\mathcal{C}$, we denote the image of s in $\mathscr{F}(Y')$ by $s|_{Y'}$. By definition, the equalizer

$$\mathrm{Eq}(\prod_{i \in I} \mathscr{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathscr{F}(U_i \times_X U_j))$$

is the set of all tuples $\{s_i \in \mathscr{F}(U_i)\}_{i \in I}$ satisfying $s_i|_{U_i \times_X U_j} = s_j|_{U_i \times_X U_j}$.

Given any tuple of elements $\{s_i \in \mathscr{F}(U_i)\}_{i \in I}$ and a map $g : V \to X$ which factors as a composition $V \xrightarrow{g_i} U_i \to X$, we can form the restriction $s_i|_V \in \mathscr{F}(V)$. In general, this will depend on the choice of i and of the map g_i. However, the condition $s_i|_{U_i \times_X U_j} = s_j|_{U_i \times_X U_j}$ is exactly what we need in order to guarantee that the restriction $s_i|_V$ is independent of i. Put another way, we have a canonical bijection

$$\mathrm{Eq}(\prod_{i \in I} \mathscr{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathscr{F}(U_i \times_X U_j)) \simeq \varprojlim_{V \in \mathcal{C}^{(0)}_{/X}} \mathscr{F}(V),$$

where $\mathcal{C}^{(0)}_{/X}$ denotes the full subcategory of $\mathcal{C}_{/X}$ spanned by those maps $V \to X$ which factor through some U_i (the factorization need not be specified). To exploit this, it will be convenient to introduce some terminology.

Definition 11. Let X be an object of $\mathcal{C}$. A *sieve on* X is a full subcategory $\mathcal{C}^{(0)}_{/X} \subseteq \mathcal{C}_{/X}$ with the property that, for each morphism $U \to V$ in $\mathcal{C}_{/X}$, if V belongs to $\mathcal{C}^{(0)}_{/X}$ then U also belongs to $\mathcal{C}^{(0)}_{/X}$. We will say that a sieve $\mathcal{C}^{(0)}_{/X}$ is *covering* if it contains a collection of maps $\{U_i \to X\}$ which form a covering of X (with respect to our chosen Grothendieck topology).

Exercise 12. Let $\{f_i : U_i \to X\}$ be a collection of morphisms in $\mathcal{C}$ having some common codomain X, and let $\mathcal{C}^{(0)}_{/X} \subseteq \mathcal{C}_{/X}$ be the full subcategory spanned by those maps $g : V \to X$ which factor through some f_i. Show that $\mathcal{C}^{(0)}_{/X}$ is a sieve on X, which is covering sieve if and only if $\{f_i : U_i \to X\}$ is a covering.

Applying the above discussion, we can reformulate the definition of sheaf as follows:

Definition 13. Let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a presheaf. Then $\mathscr{F}$ is a sheaf if and only if, for every covering sieve $\mathcal{C}^{(0)}_{/X} \subseteq \mathcal{C}_{/X}$, the map $\mathscr{F}(X) \to \varprojlim_{U \in \mathcal{C}^{(0)}_{/X}} \mathscr{F}(U)$ is a bijection.

We are now ready to describe the sheafification procedure explicitly.

Construction 14. Let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a presheaf. We define a new presheaf $\mathscr{F}^\dagger : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ by the formula

$$\mathscr{F}^\dagger(X) = \varinjlim_{\mathcal{C}^{(0)}_{/X}} \varprojlim_{U \in \mathcal{C}^{(0)}_{/X}} \mathscr{F}(U).$$

Here the colimit is taken over all covering sieves $\mathcal{C}^{(0)}_{/X}$ on X (which we regard as a partially ordered set under reverse inclusion).

Note that for any object $X \in \mathcal{C}$, the sieve $\mathcal{C}_{/X}$ is always a covering. We therefore have a canonical map

$$\alpha_{\mathscr{F}} : \mathscr{F}(X) \simeq \varprojlim_{U \in \mathcal{C}_{/X}} \mathscr{F}(U) \to \varinjlim_{\mathcal{C}^{(0)}_{/X}} \varprojlim_{U \in \mathcal{C}^{(0)}_{/X}} \mathscr{F}(U) \simeq \mathscr{F}^\dagger(X)$$

(see the proof of Lemma 18 below for a more precise definition of the presheaf $\mathscr{F}^\dagger$ and of this map). We will deduce Theorem 2 from the following more precise statements:

Proposition 15. *For every presheaf $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$, the composite map*

$$\mathscr{F} \xrightarrow{\alpha_{\mathscr{F}}} \mathscr{F}^\dagger \xrightarrow{\alpha_{\mathscr{F}^\dagger}} \mathscr{F}^{\dagger\dagger}$$

exhibits $\mathscr{F}^{\dagger\dagger}$ as a sheafification of $\mathscr{F}$. In other words, $\mathscr{F}^{\dagger\dagger}$ is a sheaf and, for every sheaf $\mathscr{G} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$, the induced map

$$\mathrm{Hom}_{\mathrm{Shv}(\mathcal{C})}(\mathscr{F}^{\dagger\dagger}, \mathscr{G}) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})}(\mathscr{F}, \mathscr{G})$$

is bijective.

Proposition 16. *The functor $\mathscr{F} \mapsto \mathscr{F}^\dagger$ is left exact: that is, it preserves finite limits.*

Let us begin with the second statement.

Lemma 17. *Let X be an object of $\mathcal{C}$. Then the collection of covering sieves $\mathcal{C}^{(0)}_{/X} \subseteq \mathcal{C}_{/X}$ is closed under finite intersections.*

Proof. Let $\mathcal{C}^{(0)}_{/X}$ and $\mathcal{C}^{(1)}_{/X}$ be covering sieves on X. Then $\mathcal{C}^{(0)}_{/X}$ contains a covering $\{U_i \to X\}_{i \in I}$ and $\mathcal{C}^{(1)}_{/X}$ contains a covering $\{V_j \to X\}_{j \in J}$. Since coverings are closed under pullback, the collection of maps $\{U_i \times_X V_j \to U_i\}_{j \in J}$ is a covering for each $i \in I$. It follows that the collection of composite maps $\{U_i \times_X V_j \to X\}$ is a covering of X. We conclude by observing that each of these maps is contained in the sieve $\mathcal{C}^{(0)}_{/X} \cap \mathcal{C}^{(1)}_{/X}$. □

Proof of Proposition 16. Fix an object $X \in \mathcal{C}$, and regard the construction

$$\mathscr{F} \mapsto \mathscr{F}^\dagger(X) = \varinjlim_{\mathcal{C}^{(0)}_{/X}} \varprojlim_{U \in \mathcal{C}^{(0)}_{/X}} \mathscr{F}(U)$$

as a functor of the presheaf $\mathscr{F}$. It follows from Lemma 17 that the colimit appearing in this expression is filtered (it is taken over a directed poset), and therefore commutes with finite limits. Since limits in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ are computed pointwise, it follows that the construction $\mathscr{F} \mapsto \mathscr{F}^\dagger$ commutes with finite limits. □

Lemma 18. *Let* $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ *be a presheaf and let* $\mathscr{G} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ *be a sheaf. Then composition with* $\alpha_{\mathscr{F}} : \mathscr{F} \to \mathscr{F}^{\dagger}$ *induces a bijection*

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S}\mathrm{et})}(\mathscr{F}^{\dagger}, \mathscr{G}) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S}\mathrm{et})}(\mathscr{F}, \mathscr{G}).$$

Proof. We begin by giving a more precise definition of the functor $\mathscr{F}^{\dagger}$ and of the natural map $\alpha : \mathscr{F} \to \mathscr{F}^{\dagger}$. For this, we need an auxiliary construction:

Notation 19. We define a category $\mathrm{Cov}(\mathcal{C})$ as follows:

- The objects of $\mathrm{Cov}(\mathcal{C})$ are pairs $(X, \mathcal{C}^{(0)}_{/X})$, where $X \in \mathcal{C}$ is an object and $\mathcal{C}^{(0)}_{/X} \subseteq \mathcal{C}_{/X}$ is a covering sieve.
- Given a pair of objects $(X, \mathcal{C}^{(0)}_{/X}), (Y, \mathcal{C}^{(0)}_{/Y}) \in \mathrm{Cov}(\mathcal{C})$, a morphism from $(X, \mathcal{C}^{(0)}_{/X})$ to $(Y, \mathcal{C}^{(0)}_{/Y})$ is a morphism $u : X \to Y$ in the category $\mathcal{C}$ with property that, for every morphism $U \to X$ belonging to the sieve $\mathcal{C}^{(0)}_{/X}$, the composite map $U \to X \xrightarrow{u} Y$ belongs to the sieve $\mathcal{C}^{(0)}_{/Y}$.

We have evident functors $i : \mathcal{C} \to \mathrm{Cov}(\mathcal{C})$ and $\rho : \mathrm{Cov}(\mathcal{C}) \to \mathcal{C}$, given by the formulae

$$i(X) = (X, \mathcal{C}_{/X}) \qquad \rho(X, \mathcal{C}^{(0)}_{/X}) = X$$

Composition with i and ρ determines pullback functors

$$i^* : \mathrm{Fun}(\mathrm{Cov}(\mathcal{C})^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \qquad \rho^* : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathrm{Fun}(\mathrm{Cov}(\mathcal{C})^{\mathrm{op}}, \mathcal{S}\mathrm{et}).$$

The functor i^* admits a right adjoint $i_* : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathrm{Fun}(\mathrm{Cov}(\mathcal{C})^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ (given by right Kan extension along i), and the functor ρ^* admits a left adjoint $\rho_! \, \mathrm{Fun}(\mathrm{Cov}(\mathcal{C})^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ (given by left Kan extension along ρ). Concretely, we have

$$i_*(\mathscr{F})(X, \mathcal{C}^{(0)}_{/X}) \simeq \varprojlim_{U \in \mathcal{C}^{(0)}_{/X}} \mathscr{F}(U) \qquad \rho_!(\mathscr{H})(X) = \varinjlim_{\mathcal{C}^{(0)}_{/X}} \mathscr{H}(X, \mathcal{C}^{(0)}_{/X}),$$

where the colimit is taken over the collection of all covering sieves on X. The construction $\mathscr{F} \mapsto \mathscr{F}^{\dagger}$ can now be described more precisely by the formula $\mathscr{F}^{\dagger} = \rho_! i_* \mathscr{F}$.

Note that, since i is a full faithful embedding, the natural map $i^* i_* \mathscr{F} \to \mathscr{F}$ is an equivalence for every presheaf $\mathscr{F}$. Consequently, for any pair of presheaves $\mathscr{F}, \mathscr{G} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$, we have a natural map

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S}\mathrm{et})}(\mathscr{F}^{\dagger}, \mathscr{G}) &= \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S}\mathrm{et})}(\rho_! i_* \mathscr{F}, \mathscr{G}) \\ &\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathrm{Cov}(\mathcal{C})^{\mathrm{op}},\mathcal{S}\mathrm{et})}(i_* \mathscr{F}, \rho^* \mathscr{G}) \\ &\xrightarrow{u} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S}\mathrm{et})}(i^* i_* \mathscr{F}, i^* \rho^* \mathscr{G}) \\ &\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S}\mathrm{et})}(\mathscr{F}, \mathscr{G}). \end{aligned}$$

These maps depends functorially on $\mathscr{G}$, and are therefore (by Yoneda's lemma) given by precomposition with some map of presheaves $\mathscr{F} \to \mathscr{F}^{\dagger}$; this gives a more precise definition of the map $\alpha_{\mathscr{F}}$ appearing in the statement of Lemma 18. To complete the proof of Lemma 18, we must show that u is a bijection when $\mathscr{G}$ is a sheaf. Note that u can be identified with the canonical map

$$\mathrm{Hom}_{\mathrm{Fun}(\mathrm{Cov}(\mathcal{C})^{\mathrm{op}},\mathcal{S}\mathrm{et})}(i_* \mathscr{F}, \rho^* \mathscr{G}) \to \mathrm{Hom}_{\mathrm{Fun}(\mathrm{Cov}(\mathcal{C})^{\mathrm{op}},\mathcal{S}\mathrm{et})}(i_* \mathscr{F}, i_* i^* \rho^* \mathscr{G}).$$

We are therefore reduced to showing that if $\mathscr{G}$ is a sheaf, then the unit map $\rho^* \mathscr{G} \to i_* i^* \rho^* \mathscr{G}$ is an equivalence of presheaves on $\mathrm{Cov}(\mathcal{C})$. Unwinding the definitions, we must show that for each object $(X, \mathcal{C}^{(0)}_{/X})$, the canonical map

$$\mathscr{G}(X) \to \varprojlim_{U \in \mathcal{C}^{(0)}_{/X}} \mathscr{G}(U)$$

is a bijection, which is a translation of our hypothesis that $\mathscr{G}$ is a sheaf (Definition 13). $\square$

To complete the proof of Proposition 15, we must show that applying the functor $\mathscr{F} \mapsto \mathscr{F}^\dagger$ twice converts any presheaf into a sheaf. This is a consequence of a more precise assertion.

Definition 20. Let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a functor. We will say that $\mathscr{F}$ is a *separated presheaf* if, for every covering $\{U_i \to X\}$, the natural map $\mathscr{F}(X) \to \prod \mathscr{F}(U_i)$ is injective. Equivalently, we say that $\mathscr{F}$ is a separated presheaf if the unit map $\rho^* \mathscr{F} \to i_* i^* \rho^* \mathscr{F}$ is a monomorphism of presheaves on $\mathrm{Cov}(\mathcal{C})$.

Notation 21. Given a presheaf $\mathscr{F}$, an object $X \in \mathcal{C}$, and a covering sieve $\mathcal{C}^{(0)}_{/X} \subseteq \mathcal{C}_{/X}$, we let $\mathscr{F}(X, \mathcal{C}^{(0)}_{/X})$ denote the value of the functor $(i_* \mathscr{F})$ on the pair $(X, \mathcal{C}^{(0)}_{/X})$, given by the direct limit $\mathscr{F}(X, \mathcal{C}^{(0)}_{/X}) = \varprojlim_{U \in \mathcal{C}^{(0)}_{/X}} \mathscr{F}(U)$.

Lemma 22. *Let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a separated presheaf. Then, for every object $X \in \mathcal{C}$ and every inclusion of covering sieves $\mathcal{C}^{(0)}_{/X} \subseteq \mathcal{C}^{(1)}_{/X} \subseteq \mathcal{C}_{/X}$, the restriction map $\mathscr{F}(X, \mathcal{C}^{(1)}_{/X}) \to \mathscr{F}(X, \mathcal{C}^{(0)}_{/X})$ is injective.*

Proof. For each object $V \in \mathcal{C}^{(1)}_{/X}$, let $\mathcal{C}^{(0)}_{/V}$ denote the sieve on V given by those maps $U \to V$ for which the composite map $U \to V \to X$ belongs to $\mathcal{C}^{(0)}_{/U}$. We then compute

$$\begin{aligned}
\mathscr{F}(X, \mathcal{C}^{(1)}_{/X}) &= \varprojlim_{V \in \mathcal{C}^{(1)}_{/X}} \mathscr{F}(V) \\
&= \varprojlim_{V \in \mathcal{C}^{(1)}_{/X}} \varprojlim_{U \in \mathcal{C}_{/V}} \mathscr{F}(U) \\
&\hookrightarrow \varprojlim_{V \in \mathcal{C}^{(1)}_{/X}} \varprojlim_{U \in \mathcal{C}^{(0)}_{/V}} \mathscr{F}(U) \\
&\simeq \varprojlim_{U \in \mathcal{C}^{(0)}_{/X}} \mathscr{F}(U).
\end{aligned}$$

□

Lemma 23. *Let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a separated presheaf. Then, for every object $X \in \mathcal{C}$ and every covering sieve $\mathcal{C}^{(0)}_{/X} \subseteq \mathcal{C}_{/X}$, the canonical map $\mathscr{F}(X, \mathcal{C}^{(0)}_{/X}) \to \mathscr{F}^\dagger(X)$ is injective.*

Proof. Apply Lemma 22 (note that a filtered colimit of injections is still an injection). □

Lemma 24. *Let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a separated presheaf. Then $\mathscr{F}^\dagger$ is a sheaf.*

Proof. Let X be an object of $\mathcal{C}$ and let $\mathcal{C}^{(1)}_{/X}$ be a covering sieve on X; we wish to show that the canonical map

$$\theta : \mathscr{F}^\dagger(X) \to \mathscr{F}^\dagger(X, \mathcal{C}^{(1)}_{/X})$$

is bijective. Write $\mathscr{F}^\dagger(X)$ as a filtered colimit $\varinjlim_{\mathcal{C}^{(0)}_{/X}} \mathscr{F}(X, \mathcal{C}^{(0)}_{/X})$. Here it suffices to take the colimit over those covering sieves $\mathcal{C}^{(0)}_{/X}$ which are contained in $\mathcal{C}^{(1)}_{/X}$. Using the notation of Lemma 22, we compute

$$\mathscr{F}(X, \mathcal{C}^{(0)}_{/X}) = \varprojlim_{V \in \mathcal{C}^{(1)}_{/X}} \mathscr{F}(V, \mathcal{C}^{(0)}_{/V}).$$

We can therefore write θ as a filtered colimit of maps

$$\theta_{\mathcal{C}^{(0)}_{/X}} : \varprojlim_{V \in \mathcal{C}^{(1)}_{/X}} \mathscr{F}(V, \mathcal{C}^{(0)}_{/V}) \to \varprojlim_{V \in \mathcal{C}^{(1)}_{/X}} \mathscr{F}^\dagger(V),$$

where $\mathcal{C}^{(0)}_{/X}$ ranges over all covering sieves on X which are contained in $\mathcal{C}^{(1)}_{/X}$. Lemma 23 guarantees that each $\theta_{\mathcal{C}^{(0)}_{/} X}$ is a monomorphism, so that θ is a monomorphism.

To complete the proof, we show that for each element $s \in \varprojlim_{V \in \mathcal{C}^{(1)}_{/X}} \mathscr{F}^{\dagger}(V)$, we can choose some covering $\mathcal{C}^{(0)}_{/X}$ for which s belongs to the image of $\theta_{\mathcal{C}^{(0)}_{/X}}$. For each object $V \in \mathcal{C}^{(1)}_{/X}$, let s_V denote the image of s in $\mathscr{F}^{\dagger}(V)$. From the definition of $\mathscr{F}^{\dagger}$, we see that there exists a covering sieve $\mathcal{C}^{(2)}_{/V} \subseteq \mathcal{C}_{/V}$ for which s_V belongs to the image of the monomorphism

$$\mathscr{F}(V, \mathcal{C}^{(2)}_{/V}) \hookrightarrow \mathscr{F}^{\dagger}(V).$$

We now complete the proof by taking $\mathcal{C}^{(0)}_{/X}$ to be the smallest sieve on X which contains each of the composite maps $U \to V \to X$, where $U \to V$ belongs to $\mathcal{C}^{(2)}_{/V}$ and $V \to X$ belongs to $\mathcal{C}^{(1)}_{/X}$. □

Lemma 25. *Let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be any presheaf. Then $\mathscr{F}^{\dagger}$ is a separated presheaf.*

Proof. We argue as in the proof of Lemma 24. Fix an object $X \in \mathcal{C}$ and let $\mathcal{C}^{(1)}_{/X}$ be a covering sieve on X; we wish to show that the canonical map

$$\theta : \mathscr{F}^{\dagger}(X) \to \mathscr{F}^{\dagger}(X, \mathcal{C}^{(1)}_{/X})$$

is injective. Write θ as as a filtered colimit of maps

$$\theta_{\mathcal{C}^{(0)}_{/X}} : \varprojlim_{V \in \mathcal{C}^{(1)}_{/X}} \mathscr{F}(V, \mathcal{C}^{(0)}_{/V}) \to \varprojlim_{V \in \mathcal{C}^{(1)}_{/X}} \mathscr{F}^{\dagger}(V),$$

where $\mathcal{C}^{(0)}_{/X}$ ranges over covering sieves on X that are contained in $\mathcal{C}^{(1)}_{/X}$. Suppose we are given a pair of elements $s, t \in \varprojlim_{V \in \mathcal{C}^{(1)}_{/X}} \mathscr{F}(V, \mathcal{C}^{(0)}_{/V})$ which have the same image under $\theta_{\mathcal{C}^{(0)}_{/X}}$. For each $V \in \mathcal{C}^{(1)}_{/X}$, let s_V and t_V denote the images of s and t in $\mathscr{F}(V, \mathcal{C}^{(0)}_{/V})$. Then s_V and t_V have the same image in $\mathscr{F}^{\dagger}(V)$. It follows that we can choose some covering sieve $\mathcal{C}^{(2)}_{/V} \subseteq \mathcal{C}^{(0)}_{/V}$ such that s_V and t_V have the same image in $\mathscr{F}(V, \mathcal{C}^{(2)}_{/V})$. Let $\mathcal{C}'_{/X}$ denote the smallest sieve on X which contains all composite maps $U \to V \to X$, where $U \to V$ belongs to $\mathcal{C}^{(2)}_{/V}$ and $V \to X$ bellongs to $\mathcal{C}^{(1)}_{/X}$. Then $\mathcal{C}'_{/X}$ is a covering sieve, and s and t have the same image in $\mathscr{F}(X, \mathcal{C}'_{/X})$, hence also in $\mathscr{F}^{\dagger}(X)$. □

Proof of Proposition 15. Combine Lemmas 18, 24, and 25. □