Lecture 7: Pretopoi

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For the reader's convenience, we recall the definition of a coherent category:

Definition 1. A category $\mathcal{C}$ is *coherent* if it satisfies the following axioms:

$(A1)$ $\mathcal{C}$ admits finite limits.

$(A2)$ Every morphism $f : X \to Z$ factors as a composition $X \xrightarrow{g} Y \xrightarrow{h} Z$, where h is a monomorphism and g is an effective epimorphism.

$(A3)$ For every object $X \in \mathcal{C}$, the poset $\mathrm{Sub}(X)$ is an upper semilattice: that is, it has a least element and joins $X_0 \vee X_1$.

$(A4)$ The collection of effective epimorphisms in $\mathcal{C}$ is stable under pullbacks.

$(A5)$ For every morphism $f : X \to Y$ in $\mathcal{C}$, the map $f^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ is a homomorphism of upper semilattices: that is, it preserves least elements and joins.

Let T be a first-order theory. In Lecture 2, we associated to T a category $\mathrm{Syn}_0(T)$, which we call the *weak syntactic category of* T. However, there is a sense in which this category is not really an invariant of T. More precisely, there are examples of first-order theories that we might like to say are equivalent, despite the fact that their weak syntactic categories are not equivalent (as categories).

Example 2. We define first-order theories T and T' as follows:

- The language of T has no predicates, and T has a single axiom $(\exists! x)[x = x]$.
- The language of T' has a single 1-ary predicate P, and a pair of axioms

$$(\exists! y)[P(y)] \qquad (\exists! z)[\neg P(z)].$$

Up to isomorphism, the theory T has only one model: a set M having exactly one element. Similarly, the theory T' has only one model, consisting of a pair of distinct points (one of which satisfies the predicate P, and one of which does not). In particular, the categories $\mathrm{Mod}(T)$ and $\mathrm{Mod}(T')$ are equivalent. However, the weak syntactic categories $\mathrm{Syn}_0(T)$ and $\mathrm{Syn}_0(T')$ are not equivalent.

Exercise 3. Show that the category $\mathrm{Syn}_0(T)$ is equivalent to the poset $\{0 < 1\}$, and that $\mathrm{Syn}_0(T')$ is equivalent to the category of finite sets $\mathcal{S}\mathrm{et}^{\mathrm{fin}}$.

Our ultimate goal in this course is to prove Makkai's strong conceptual completeness theorem, which asserts roughly that one can recover the syntax of a first-order theory T from its semantics (encoded by the category of models $\mathrm{Mod}(T)$ with the structure given by ultraproducts). To have any hope of proving such a statement, we cannot interpret "the syntax of T" as being its weak syntactic category $\mathrm{Syn}_0(T)$: Example 2 shows that two theories can have essentially the same semantics, but different weak syntactic categories. We will correct this issue by replacing $\mathrm{Syn}_0(T)$ by a certain enlargement, which we will denote by $\mathrm{Syn}(T)$ and refer to as the *syntactic category of* T.

Before defining this enlargement in general, let's begin by inspecting the difference between the categories $\mathrm{Syn}_0(T)$ and $\mathrm{Syn}_0(T')$ appearing in Example 2. Both of these categories are coherent: in particular, they are categories in which we can form unions of subobjects of a fixed object. However, in the category $\mathrm{Syn}_0(T') \simeq \mathcal{S}\mathrm{et}^{\mathrm{fin}}$ we can do something a little bit better: given two finite sets S and T we can form their *disjoint union* $S \amalg T$. This is a new finite set which contains S and T as (disjoint) subobjects. Categorically, it can be described as coproduct of S and T in the category $\mathcal{S}\mathrm{et}^{\mathrm{fin}}$. This is actually an instance of a general phenomenon:

Proposition 4. *Let $\mathcal{C}$ be a coherent category (or, more generally, any category satisfying $(A1)$, $(A3)$, and $(A5)$). Let X be an object of $\mathcal{C}$ and suppose we are given a collection of subobjects $X_1, X_2, \ldots, X_n \subseteq X$, such that $X_i \wedge X_j = \emptyset$ for $i \neq j$ (here $\emptyset$ denotes the least element of $\mathrm{Sub}(X)$). Then $X_1 \vee \cdots \vee X_n$ is a coproduct of the subobjects $\{X_i\}_{1 \leq i \leq n}$ in the category $\mathcal{C}$.*

Proof. Without loss of generality, we may assume that $X = X_1 \vee \cdots \vee X_n$. Suppose we are given an object $Y \in \mathcal{C}$ and a collection of maps $f_i : X_i \to Y$. We wish to show that there is a unique map $f : X \to Y$ satisfying $f|_{X_i} = f_i$. For $1 \leq i \leq n$, regard $\Gamma(f_i)$ as a subobject of $X_i \times Y \subseteq X \times Y$, and set $Z = \Gamma(f_1) \vee \cdots \vee \Gamma(f_n)$. We will complete the proof by showing that Z is the graph of map from X to Y: that is, that the composition $Z \hookrightarrow X \times Y \to X$ is an isomorphism. Let us denote the composition by h.

We first claim that h is a monomorphism. To prove this, we note that $Z \times_X Z$ can be identified with a subobject of the product $(X \times Y) \times_X (X \times Y) \simeq X \times Y \times Y$. Let $\pi : X \times Y \times Y \to X$ be the projection map. Using axiom $(A5)$, we can identify $Z \times_X Z$ with the join of the subobjects $\Gamma(f_i) \times_X \Gamma(f_j)$. For $i \neq j$, we have $\Gamma(f_i) \times_X \Gamma(f_j) \subseteq \pi^{-1}(X_i \wedge X_j) = \pi^{-1}(\emptyset)$. Since π^{-1} preserves least elements, it follows that $\Gamma(f_i) \times_X \Gamma(f_j)$ is a smallest element of $\mathrm{Sub}(X \times Y \times Y)$. It follows that $Z \times_X Z$ is given by the join of the subobjects $\Gamma(f_i) \times_X \Gamma(f_i)$, each of which is contained in the image of the diagonal $Z \to Z \times_X Z$. We therefore have $Z \simeq Z \times_X Z$, so that h is a monomorphism as desired.

Let $\mathrm{Im}(h)$ denote the subobject of X determined by the monomorphism $h : Z \to X$. Since Z contains each $\Gamma(f_i)$, we have $X_i \subseteq \mathrm{Im}(h)$ for $1 \leq i \leq n$. The equality $X = X_1 \vee \cdots \vee X_n$ then implies that $\mathrm{Im}(h) = X$, so that h is an isomorphism as desired. □

In the special case $n = 0$, we obtain the following:

Corollary 5. *Let $\mathcal{C}$ be a coherent category. For any object $X \in \mathcal{C}$, the least element of $\mathrm{Sub}(X)$ is initial when regarded as an object of $\mathcal{C}$. In particular, $\mathcal{C}$ has an initial object, which we will henceforth denote by $\emptyset$.*

Definition 6. Let $\mathcal{C}$ be a category which admits fiber products, and let $X, Y \in \mathcal{C}$ be a pair of objects which admits a coproduct $X \amalg Y$. We will say that $X \amalg Y$ is a *disjoint coproduct* of X and Y if the following pair of conditions is satisfied:

- Each of the maps $X \to (X \amalg Y) \leftarrow Y$ is a monomorphism.
- The fiber product $X \times_{X \amalg Y} Y$ is an initial object of $\mathcal{C}$.

We will that $\mathcal{C}$ *has disjoint coproducts* if it has an initial object and every pair of objects $X, Y \in \mathcal{C}$ has a disjoint coproduct $X \amalg Y$.

Corollary 7. *Let $\mathcal{C}$ be a coherent category containing an object X and let $X_0, X_1 \subseteq X$ be subobjects satisfying $X_0 \wedge X_1 = \emptyset$. Then $X_0 \vee X_1$ is a disjoint coproduct of X_0 and X_1.*

Proof. We have monomorphisms $X_0 \hookrightarrow X_0 \vee X_1 \hookleftarrow X_1$, and the fiber product

$$X_0 \times_{X_0 \vee X_1} X_1 \simeq X_0 \times_X X_1 \simeq \emptyset$$

is an initial object of $\mathcal{C}$ by assumption. Proposition 4 guarantees that these maps exhibit $X_0 \vee X_1$ as a coproduct of X_0 and X_1. □

Proposition 8. *Let $\mathcal{C}$ be a category. The following conditions are equivalent:*

(1) *The category $\mathcal{C}$ is coherent and has disjoint coproducts.*

(2) *The category $\mathcal{C}$ satisfies $(A1)$, $(A2)$, and $(A4)$, along with the following modified versions of $(A3)$ and $(A5)$:*

   ($A3'$) *The category $\mathcal{C}$ has disjoint coproducts.*

   ($A5'$) *The formation of finite coproducts in $\mathcal{C}$ is preserved by pullbacks. More precisely, for every morphism $f : X \to Y$ in $\mathcal{C}$, the pullback functor*

$$f^* : \mathcal{C}_{/Y} \to \mathcal{C}_{/X} \qquad f^*(U) = U \times_Y X$$

   preserves finite coproducts.

Proof. Suppose first that (1) is satisfied. Then $(A1)$, $(A2)$, $(A3')$, and $(A4)$ are automatic. We will prove $(A5')$. Fix a morphism $f : X \to Y$ in $\mathcal{C}$ and a collection of objects $\{Y_i\}_{1 \leq i \leq n}$ in $\mathcal{C}_{/Y}$. We wish to show that the canonical map

$$(X \times_Y Y_1) \amalg \cdots \amalg (X \times_Y Y_n) \to X \times_Y (Y_1 \amalg \cdots \amalg Y_n)$$

is an isomorphism. To prove this, we can replace Y by the coproduct $Y_1 \amalg \cdots \amalg Y_n$ (and X by the fiber product $X \times_Y (Y_1 \amalg \cdots \amalg Y_n)$), and thereby reduce to the case where Y is the coproduct of the objects Y_i. Since coproducts in $\mathcal{C}$ are disjoint, we can view each Y_i as a subobject of Y, and these subobjects are disjoint (that is, $Y_i \wedge Y_j = \emptyset$ for $i \neq j$). Similarly, we can view the fiber products $X \times_Y Y_i$ as disjoint subobjects of X. In this case, Proposition 4 supplies identifications

$$Y_1 \amalg \cdots \amalg Y_n \simeq Y_1 \vee \cdots \vee Y_n$$

$$(X \times_Y Y_1) \amalg \cdots \amalg (X \times_Y Y_n) \simeq (X \times_Y Y_1) \vee \cdots \vee (X \times_Y Y_n)$$

(where the joins are formed in $\mathrm{Sub}(Y)$ and $\mathrm{Sub}(X \times Y)$, respectively). We are therefore reduced to showing that the map f^{-1} preserves joins of subobjects, which is a special case of axiom $(A5)$.

We now show that (2) $\Rightarrow$ (1). Assume that the category $\mathcal{C}$ satisfies $(A1)$, $(A2)$, $(A3')$, $(A4)$, and $(A5')$. We wish to show that it is a coherent category: that is, it satisfies $(A3)$ and $(A5)$. Let X be an object of $\mathcal{C}$, and suppose we are given a collection of subobjects $X_1, \ldots, X_n \subseteq X$; we wish to show that there exists a least upper bound U for the set $\{X_1, \ldots, X_n\}$ in the poset $\mathrm{Sub}(X)$. Assumption $(A3')$ guarantees that there exists a coproduct $X_1 \amalg \cdots \amalg X_n$ in $\mathcal{C}$. The least upper bound U is then given by taking U to be the image of the map $X_1 \amalg \cdots \amalg X_n \to X$ (which exists by virtue of $(A2)$). This completes the proof of $(A3)$. To prove $(A5)$,we must show that the formation of least upper bounds is compatible with pullback along a morphism $f : Y \to X$. This follows from the construction, since coproducts are compatible with pullback $(A5')$ and images are compatible with pullback by $(A4)$. □

We now discuss another sort of construction that we cannot quite carry out in a coherent category.

Definition 9. Let $\mathcal{C}$ be a category which admits finite limits and let X be an object of $\mathcal{C}$. We say that a subobject $R \subseteq X \times X$ is an *equivalence relation* on X if, for every object $Y \in \mathcal{C}$, the image of the induced map

$$\mathrm{Hom}_{\mathcal{C}}(Y, R) \to \mathrm{Hom}_{\mathcal{C}}(Y, X \times X) \simeq \mathrm{Hom}_{\mathcal{C}}(Y, X) \times \mathrm{Hom}_{\mathcal{C}}(Y, X)$$

is an equivalence relation on the set $\mathrm{Hom}_{\mathcal{C}}(Y, X)$.

Every morphism $f : X \to Y$ determines an equivalence relation R on X, given by the fiber product $R = X \times_Y X$. If f is an effective epimorphism, then we can recover the object Y (and the morphism f) from the equivalence relation R: namely, it is the coequalizer of the pair of projection maps $\pi, \pi' : X \times_Y X \to X$.

Definition 10. Let $\mathcal{C}$ be a category which admits finite limits, let X be an object of $\mathcal{C}$, and let $R \subseteq X \times X$ be an equivalence relation. We say that R is *effective* if there exists an effective epimorphism $f : X \to Y$ such that $R = X \times_Y X$ (as subobject of $X \times X$).

Example 11. In the category of sets, every equivalence relation is effective. This is just the statement that if $R \subseteq X \times X$ is an equivalence relation on X, then an ordered pair $(x, y) \in X \times X$ belongs to R if and only if x and y have the same image in the quotient X/R.

Proposition 12. *Let $\mathcal{C}$ be a category satisfying $(A1)$ and $(A4)$. Suppose that every equivalence relation in $\mathcal{C}$ is effective. Then $\mathcal{C}$ also satisfies $(A2)$.*

The proof will require an elementary observation:

Lemma 13. *Let $\mathcal{C}$ be a category satisfying $(A1)$ and $(A4)$ and suppose we are given a pullback diagram*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

in $\mathcal{C}$. If f is an effective epimorphism and g' is an isomorphism, then g is an isomorphism.

Proof. Using $(A4)$ we see that f' is also an effective epimorphism. We have a commutative diagram

$$\begin{array}{ccccc} X' \times_{Y'} X' & \rightrightarrows & X' & \longrightarrow & Y' \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle g} \\ X \times_Y X & \rightrightarrows & X & \longrightarrow & Y \end{array}$$

where the rows are coequalizer diagrams and the left and middle vertical maps are isomorphisms. It follows that g is an isomorphism as well. □

Proof of Proposition 12. Let $f : X \to Z$ be a morphism in $\mathcal{C}$. Then $R = X \times_Z X$ is an equivalence relation on X. Suppose that this equivalence relation is effective: that is, there exists an effective epimorphism $g : X \to Y$ such that $R = X \times_Y X$ (as subobjects of $X \times X$). In particular, Y is the coequalizer of the projection maps $\pi, \pi' : R \to X$. Since $f \circ \pi = f \circ \pi'$, there is a unique morphism $h : Y \to Z$ such that $f = h \circ g$. To complete the proof, it will suffice to show that h is a monomorphism: that is, that the diagonal map $Y \to Y \times_Z Y$ is an isomorphism. We have a commutative diagram of pullback squares

$$\begin{array}{ccccc} X \times_Y X & \xrightarrow{\delta'} & X \times_Z X & \longrightarrow & X \times X \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle g \times g} \\ Y & \xrightarrow{\delta} & Y \times_Z Y & \longrightarrow & Y \times Y. \end{array}$$

Since g is an effective epimorphism, the vertical maps in this diagram are effective epimorphisms. Moreover, δ' is an isomorphism by construction (since $X \times_Z X$ and $X \times_Y X$ are both equal to R as subobjects of $X \times X$). Applying Lemma 13, we see that δ is an isomorphism, as desired. □

We are now ready to introduce one of the central objects of interest in this course.

Definition 14. Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is a *pretopos* if it satisfies the following axioms:

$(A1)$ The category $\mathcal{C}$ admits finite limits.

$(A2')$ Every equivalence relation in $\mathcal{C}$ is effective.

$(A3')$ The category $\mathcal{C}$ admits finite coproducts, and coproducts are disjoint.

$(A4)$ The collection of effective epimorphisms in $\mathcal{C}$ is closed under pullbacks.

($A5'$) The formation of finite coproducts in $\mathcal{C}$ is preserved by pullback.

We have proven:

Proposition 15. *Every pretopos is a coherent category.*

Proof. Combine Propositions 12 and 8. ☐

Example 16. The category of sets is a pretopos.

Example 17. The category of finite sets is a pretopos.