Lecture 24X-Ultracategory Fibrations

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Let $\mathcal{C}$ be a small pretopos, which we regard as fixed throughout this lecture. Recall that an object of $\mathrm{Stone}_{\mathcal{C}}$ is *free* if it can be written as a coproduct $\coprod_{i\in I}(\{i\}, M_i)$, where $\{M_i\}_{i\in I}$ is a collection of models of $\mathcal{C}$ indexed by a set I. Let $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}$ denote the full subcategory of $\mathrm{Stone}_{\mathcal{C}}$ spanned by the free objects. By definition, the category $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}$ contains the category of models $\mathrm{Mod}(\mathcal{C})^{\mathrm{op}}$ as a full subcategory, and every object of $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}$ can be written as a coproduct of objects of $\mathrm{Mod}(\mathcal{C})^{\mathrm{op}}$. Our goal in this lecture is to address the following:

Question 1. What is the structure of the category $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}$? To what extent can it be reconstructed from the category $\mathrm{Mod}(\mathcal{C})$ of models of $\mathcal{C}$?

Let us begin by treating the case where $\mathcal{C} = \mathcal{S}\mathrm{et}_{\mathrm{fin}}$ is the category of finite sets. Then $\mathrm{Stone}_{\mathcal{C}}$ can be identified with the category Stone of Stone spaces, and $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}$ can be identified with the category $\mathrm{Stone}^{\mathrm{fr}}$ of *free* Stone spaces: that is, Stone spaces which have the form βI, for some set I. This category can be described explicitly as follows:

- The objects of $\mathrm{Stone}^{\mathrm{fr}}$ are Stone spaces of the form βI, where I is a set.
- Morphisms in $\mathrm{Stone}^{\mathrm{fr}}$ are given by continuous maps of Stone spaces $\beta I \to \beta J$. Using the universal property of βI, we see that a morphism from βI to βJ is just given by a map of sets $I \to \beta J$, or equivalently a collection $\{\mathcal{U}_i\}_{i\in I}$ of ultrafilters on J which parametrized by the set I.

Remark 2. The construction $I \mapsto \beta I$ determines a faithful and essentially surjective functor from the category $\mathcal{S}\mathrm{et}$ to the category $\mathrm{Stone}^{\mathrm{fr}}$. For every pair of sets I and J, we can identify $\mathrm{Hom}_{\mathcal{S}\mathrm{et}}(I, J)$ with the subset of $\mathrm{Hom}_{\mathrm{Stone}^{\mathrm{fr}}}(\beta I, \beta J) \simeq \mathrm{Hom}_{\mathcal{S}\mathrm{et}}(I, \beta J)$ consisting of those maps which carry I into J. In other words, it corresponds to the subset of $\mathrm{Hom}_{\mathrm{Stone}^{\mathrm{fr}}}(\beta I, \beta J)$ corresponding to those collections of ultrafilters $\{\mathcal{U}_i\}_{i\in I}$ where each $\mathcal{U}_i$ is principal.

Exercise 3. Let I, J, and K be sets, and suppose we are given morphisms

$$\beta I \xrightarrow{f} \beta J \xrightarrow{g} \beta K,$$

so that f determines a collection $\{\mathcal{U}_i\}_{i\in I}$ of ultrafilters on J, and g determiens a collection $\{\mathcal{V}_j\}_{j\in J}$ of ultrafilters on K. Show that the composition $g \circ f$ corresponds to the collection of ultrafilters $\{\mathcal{W}_i\}_{i\in I}$ on K, where

$$(K_0 \in \mathcal{W}_i) \Leftrightarrow (\{j \in J : K_0 \in \mathcal{V}_j\} \in \mathcal{U}_i).$$

Let's now return to the case of a general pretopos $\mathcal{C}$. Note that we have a forgetful functor

$$\pi : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}} \to \mathrm{Stone}^{\mathrm{fr}} \qquad \pi(X, \mathcal{O}_X) = X.$$

We now articulate a special feature of the functor π.

Definition 4. Let $\pi : \mathcal{E} \to \mathcal{B}$ be a functor between categories. For each object $B \in \mathcal{B}$, we let $\mathcal{E}_B$ denote the fiber product $\mathcal{E} \times_{\mathcal{B}} \{B\}$.

Let $\overline{f} : E' \to E$ be a morphism in $\mathcal{E}$ having image $f : B' \to B$ in the category $\mathcal{B}$. We will say that $\overline{f}$ is *locally π-Cartesian* if the following condition is satisfied: for every object $E'' \in \mathcal{E}_{B'}$ with composition with $\overline{f}$ induces a bijection

$$\mathrm{Hom}_{\mathcal{E}_{B'}}(E'', E') \to \mathrm{Hom}_{\mathcal{E}}(E'', E) \times_{\mathrm{Hom}_{\mathcal{B}}(B', B)} \{f\}.$$

We say that π is a *local Grothendieck fibration* if, for every object $E \in \mathcal{E}$ and every morphism $f : B' \to \pi(E)$ in $\mathcal{B}$, there exists a locally π-Cartesian morphism $\overline{f} : E' \to E$ with $\pi(f) = f$. In this case, it follows from Yoneda's lemma that the object E' is well-defined up to canonical isomorphism as an object of the category $\mathcal{E}_{B'}$. Moreover, the construction $E \mapsto E'$ determines a functor $f^* : \mathcal{E}_B \to \mathcal{E}_{B'}$.

Remark 5. Let $\pi : \mathcal{E} \to \mathcal{B}$ be a local Grothendieck fibration of categories, and suppose we are given morphisms

$$B'' \xrightarrow{g} B' \xrightarrow{f} B$$

in the category $\mathcal{B}$. For each object $E \in \mathcal{E}_B$, we can lift f to a locally π-Cartesian morphism $\overline{f} : f^*E \to E$, and we can lift g to a locally π-Cartesian morphism $\overline{g} : g^*f^*E \to f^*E$. The composition $(\overline{f} \circ \overline{g}) : g^*f^*E \to E$ induces a comparison map $\rho_E : g^*f^*E \to (f \circ g)^*E$. This construction depends functorially on E, and therefore determines a natural transformation of functors $\rho : g^* \circ f^* \to (f \circ g)^*$.

The following conditions on π are equivalent:

- The collection of locally π-Cartesian morphisms of $\mathcal{E}$ is closed under composition.
- For every pair of morphisms $B'' \xrightarrow{g} B' \xrightarrow{f} B$ in $\mathcal{B}$, the comparison map $g^* \circ f^* \to (f \circ g)^*$ is an isomorphism.

If these conditions are satisfied, we say that π is a *Grothendieck fibration.*

Proposition 6. *The forgetful functor $\pi : \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}} \to \mathrm{Stone}^{\mathrm{fr}}$ is a local Grothendieck fibration.*

Proof. Suppose we are given an object $(X, \mathcal{O}_X) \in \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}$ and a morphism $f : Y \to X$ in $\mathrm{Stone}^{\mathrm{fr}}$; we wish to lift f to a locally π-Cartesian morphism

$$\overline{f} : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

in $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}$. Write $Y = \beta I$ for some set I, so that f determines a map from I to X (which we will also denote by f). We take $(Y, \mathcal{O}_Y)$ to be the coproduct $\coprod_{i \in I}(\{i\}, \mathcal{O}_{X,f(i)})$. The canonical maps $(\{i\}, \mathcal{O}_{X,f(i)}) \to (X, \mathcal{O}_X)$ then amalgamate to a map $\overline{f} : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$. We claim that $\overline{f}$ is locally π-Cartesian. To prove this, we must show that for any other object $(Y, \mathcal{O}'_Y) \in \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}$, the canonical map

$$\theta : \mathrm{Hom}_{\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}}((Y, \mathcal{O}'_Y), (Y, \mathcal{O}_Y)) \times_{\mathrm{Hom}_{\mathrm{Stone}}(Y,Y)} \{\mathrm{id}_Y\} \to \mathrm{Hom}_{\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}}}((Y, \mathcal{O}'_Y), (X, \mathcal{O}_X)) \times_{\mathrm{Hom}_{\mathrm{Stone}}(Y,X)} \{f\}$$

is bijective. Since $(Y, \mathcal{O}'_Y)$ is free, we can write it as a coproduct $\coprod_{i \in I}(\{i\}, M_i)$ for some collection of models $\{M_i\}_{i \in I}$. We can then factor θ as a product of maps

$$\theta_i : \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(\mathcal{O}_{Y,i}, M_i) \to \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(\mathcal{O}_{X,f(i)}, M_i).$$

Each of these maps is bijective, because $\overline{f}$ induces an isomorphism of models $\mathcal{O}_{X,f(i)} \simeq \mathcal{O}_{Y,i}$. □

For each object $X \in \mathrm{Stone}^{\mathrm{fr}}$, let $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C},X}$ denote the fiber product $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}} \times_{\mathrm{Stone}^{\mathrm{fr}}} \{X\}$. It follows from Proposition 6 that every continuous map $f : X \to Y$ in $\mathrm{Stone}^{\mathrm{fr}}$ induces a pullback functor $f^* : \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C},Y} \to \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C},X}$.

Proposition 7. *Let I be a set and set $X = \beta I$. For each $i \in I$, let $f_i : \{i\} \hookrightarrow X$ be the inclusion map. Then the pullback functors f_i^* induce an equivalence of categories*

$$\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C},X} \to \prod_{i \in I} \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C},\{i\}} \simeq \prod_{i \in I} \mathrm{Mod}(\mathcal{C})^{\mathrm{op}}.$$

Proof. Concretely, this functor is given by

$$((X, \mathcal{O}_X) \in \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C},X}) \mapsto \{\mathcal{O}_{X,i}\}_{i \in I}.$$

An inverse functor is given by $\{M_i\}_{i \in I} \mapsto \coprod_{i \in I}(\{i\}, M_i)$. □

Remark 8. In what follows, it will be convenient to use Proposition 7 to identify each of the categories $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C},\beta I}$ with $(\mathrm{Mod}(\mathcal{C})^{\mathrm{op}})^I$. Suppose that $f : \beta I \to \beta J$ is a continuous map of Stone spaces, corresponding to a family of ultrafilters $\{\mathcal{U}_i\}_{i \in I}$ on the set J. Then the associated pullback functor

$$f^* : (\mathrm{Mod}(\mathcal{C})^{\mathrm{op}})^J \simeq \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C},\beta J} \to \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C},\beta I} \simeq (\mathrm{Mod}(\mathcal{C})^{\mathrm{op}})^I$$

is given by the construction

$$\{M_j\}_{j \in J} \mapsto \{(\prod_{j \in J} M_j)/\,\mathcal{U}_i\}_{i \in I}.$$

In other words, the local Grothendieck fibration π encodes the operation of "forming ultraproducts in $\mathrm{Mod}(\mathcal{C})$".

Warning 9. The local Grothendieck fibration $\pi : \mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}} \to \mathrm{Stone}^{\mathrm{fr}}$ is usually not a Grothendieck fibration. To see this, consider a pair of composable morphisms $\beta I \xrightarrow{g} \beta J \xrightarrow{f} \beta K$ in the category $\mathrm{Stone}^{\mathrm{fr}}$, corresponding to a family of ultrafilters $\{\mathcal{U}_i\}_{i \in I}$ on the set J and a family of ultrafilters $\{\mathcal{V}_j\}_{j \in J}$ on the set K. The composition $f \circ g$ then corresponds to a family of ultrafilters $\{\mathcal{W}_i\}_{i \in I}$ on the set K, as described in Exercise 3. We then have a natural transformation $g^* \circ f^* \to (f \circ g)^*$ of functors from $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C},\beta K} \simeq (\mathrm{Mod}(\mathcal{C})^{\mathrm{op}})^K$ to $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C},\beta I} \simeq (\mathrm{Mod}(\mathcal{C})^{\mathrm{op}})^I$. To a collection of models $\{M_k\}_{k \in K}$, this natural transformation associates a collection of maps

$$\rho_i : (\prod_{k \in K} M_k)/\,\mathcal{W}_i \to (\prod_{j \in J}((\prod_{k \in K} M_k)/\,\mathcal{V}_j))/\,\mathcal{U}_i\,.$$

of models of $\mathcal{C}$. These maps are usually not isomorphisms. For example, suppose we are given an object $C \in \mathcal{C}$ for which each $M_k(C)$ is nonempty. In this case, we obtain a map of sets

$$\rho_i(C) : (\prod_{k \in K} M_k(C))/\,\mathcal{W}_i \to (\prod_{j \in J}((\prod_{k \in K} M_k(C))/\,\mathcal{V}_j))/\,\mathcal{U}_i\,.$$

whose domain can be identified with a quotient of the product $\prod_{k \in K} M_k(C)$, and whose codomain can be identified with a quotient of the product $\prod_{k \in K} M_k(C)^J$. This map is injective but usually not surjective.

Example 10. In the situation of Warning 9, suppose that we take I and K to be one-element sets, so that $f : \beta J \to \beta K$ is uniquely determined and $g : \beta I \to \beta J$ is given by specifying an ultrafilter $\mathcal{U}$ on the set J. Let M be a model of $\mathcal{C}$, which we can identify with an object of $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C},\beta K}$. In this case, we can identify $(f \circ g)^* M = \mathrm{id}^* M$ with M, and $(g^* \circ f^*)(M)$ with the ultrapower $M^J/\,\mathcal{U}$. Under these identifications, the natural transformation $g^* \circ f^* \to (f \circ g)^*$ induces the diagonal embedding

$$\delta_M : M \hookrightarrow M^J/\,\mathcal{U}\,.$$

We now describe a situation where the issue raised in Warning 9 does not arise.

Proposition 11. *Let $g : \beta I \to \beta J$ and $f : \beta J \to \beta K$ be maps in* $\mathrm{Stone}^{\mathrm{fr}}$*, and suppose that g carries I into J (that is, g arises from a map of sets $I \to J$). Then the natural transformation $g^* \circ f^* \to (f \circ g)^*$ is an equivalence of functors from* $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C},\beta K}$ *to* $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}\,\beta I}$.

Proof. Let us identify g and f with collections of ultrafilters $\{\mathcal{U}_i\}_{i \in I}$ on the set J and $\{\mathcal{V}_j\}_{j \in J}$ on the set K, respectively. Our assumption is that each $\mathcal{U}_i$ is the principal ultrafilter associated to some element $g(i) \in J$. In this case, the composite map $f \circ g : \beta I \to \beta K$ corresponds to the family of ultrafilters $\{\mathcal{V}_{g(i)}\}_{i \in I}$ on K. The desired result now follows from the observation that for any collection of sets $\{S_k\}_{k \in K}$, the canonical map

$$(\prod_{k \in K} S_k)/\mathcal{V}_{g(i)} \to (\prod_{j \in J}((\prod_{k \in K} S_k)/\mathcal{V}_j))/\mathcal{U}_i$$

is bijective (this is immediate from our assumption that $\mathcal{U}_i$ is principal). □

We now propose the following preliminary answer to Question 1:

Definition 12. An *ultracategory fibration* is a category $\mathcal{E}$ together with a functor $\pi : \mathcal{E} \to \mathrm{Stone}^{\mathrm{fr}}$ with the following properties:

(1) The functor π is a local Grothendieck fibration.

(2) Let I be a set and let $f_i : \{i\} \hookrightarrow \beta I$ denote the inclusion map for each $i \in I$. Then the construction

$$(M \in \mathcal{E}_{\beta I}) \mapsto \{f_i^* M \in \mathcal{E}_{\{i\}}\}_{i \in I}$$

induces an equivalence of categories

$$\mathcal{E}_{\beta I} \to \prod_{i \in I} \mathcal{E}_{\{i\}}.$$

(3) Let $g : \beta I \to \beta J$ and $f : \beta J \to \beta K$ be maps in $\mathrm{Stone}^{\mathrm{fr}}$, and suppose that g carries I into J. Then the natural transformation $g^* \circ f^* \to (f \circ g)^*$ is an equivalence of functors from $\mathcal{M}_{\beta K}$ to $\mathcal{E}_{\beta I}$.

Example 13. Let $\mathcal{C}$ be a small pretopos. Then the forgetful functor $\mathrm{Stone}^{\mathrm{fr}}_{\mathcal{C}} \to \mathrm{Stone}^{\mathrm{fr}}$ is an ultracategory fibration.