Lecture 22X-Embedding into an Ultrapower

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Let $\mathcal{C}$ be a small pretopos, which we regard as fixed through this lecture. We let $\mathrm{Pro}^{\mathrm{fr}}(\mathcal{C})$ denote the full subcategory of $\mathrm{Pro}(\mathcal{C})$ spanned by the *free* objects (that is, those which can be written as coproducts of models), and we define $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}} \subseteq \mathrm{Stone}_{\mathcal{C}}$ similarly, so that we have an equivalence $\mathrm{Pro}^{\mathrm{fr}}(\mathcal{C}) \simeq \mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}$. We have fully faithful embeddings

$$\mathcal{C} \hookrightarrow \mathrm{Shv}(\mathcal{C}) \hookrightarrow \mathrm{Shv}(\mathrm{Pro}(\mathcal{C})) \simeq \mathrm{Shv}(\mathrm{Pro}^{\mathrm{fr}}(\mathcal{C})) \simeq \mathrm{Shv}(\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}) \subseteq \mathrm{Fun}(\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr,op}}, \mathcal{S}\mathrm{et}).$$

We have seen that every functor $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr,op}} \to \mathcal{S}\mathrm{et}$ which belongs to the essential image of the composite embedding must satisfy the following pair of conditions:

(a') For every collection of models $\{M_i\}_{i \in I}$ of $\mathcal{C}$, the canonical map

$$\mathscr{F}(\coprod_{i \in I}(\{i\}, M_i)) \to \prod_{i \in I} \mathscr{F}(\{i\}, M_i)$$

is a bijection.

(b') For every object $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}$ and every point $x \in X$, the canonical map $\varinjlim_{x \in U} \mathscr{F}(U, \mathcal{O}_X|_U) \to \mathscr{F}(\{x\}, \mathcal{O}_{X,x})$ is a bijection. Here U ranges over all clopen neighborhoods of x in X.

Remark 1. Suppose that $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr,op}} \to \mathcal{S}\mathrm{et}$ satisfies conditions (a') and (b'). Let $\{M_i\}_{i \in I}$ be a collection of models of M, and set $(X, \mathcal{O}_X) = \coprod_{i \in I}(\{i\}, M_i)$ in $\mathrm{Stone}_{\mathcal{C}}$. Suppose that x is a point of X, corresponding to an ultrafilter $\mathcal{U}$ on the set I. Then every clopen neighborhood of x in X has the form $U_J = \{\mathcal{U}' \in X : J \in \mathcal{U}'\}$, where J belongs to the ultrafilter $\mathcal{U}$. We then compute

$$\begin{aligned}
\mathscr{F}((\prod_{i \in I} M_i)/\mathcal{U}) &\simeq \mathscr{F}(\{x\}, \mathcal{O}_{X,x}) \\
&\simeq \varinjlim_{x \in U_J} \mathscr{F}(U_J, \mathcal{O}_X|_{U_J}) \\
&\simeq \varinjlim_{J \in \mathcal{U}} \mathscr{F}(\coprod_{i \in J}(\{i\}, M_i)) \\
&\simeq \varinjlim_{J \in \mathcal{U}} \prod_{i \in J} \mathscr{F}(\{i\}, M_i) \\
&\simeq (\prod_{i \in I} \mathscr{F}(M_i))/\mathcal{U}.
\end{aligned}$$

In other words, the functor $\mathscr{F}|_{\mathrm{Mod}(\mathcal{C})}$ "commutes with ultraproducts". We will return to this observation later.

Remark 2. Since every object of $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}$ can be written as a coproduct of models, condition (a') is equivalent to the following *a priori* stronger condition:

(a'') The functor $\mathscr{F}$ carries coproducts in $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}$ to products in $\mathcal{S}\mathrm{et}$.

Over the next few lectures, we will prove the converse: any functor $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr,op}} \to \mathcal{S}\mathrm{et}$ satisfying (a') and (b') arises from an object of $\mathcal{C}$. In this lecture, we will carry out the first step by proving the following:

Theorem 3. *Let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr,op}} \to \mathcal{S}\mathrm{et}$ be a functor satisfying (a') and (b'). Then $\mathscr{F}$ also satisfies the following:*

(d) *For every elementary morphism $f : M \to N$ in $\mathrm{Mod}(\mathcal{C})$, we have an equalizer diagram*

$$\mathscr{F}(M) \to \mathscr{F}(N) \rightrightarrows \prod \mathscr{F}(P)$$

where the product is taken over all commutative diagrams $M \xrightarrow{f} N \rightrightarrows P$.

Corollary 4. *Let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr,op}} \to \mathcal{S}\mathrm{et}$ be a functor satisfying (a') and (b'). Then $\mathscr{F}$ is a sheaf (with respect to the topology on $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}$ generated by the finite coverings in $\mathrm{Stone}_{\mathcal{C}}$).*

Proof. This follows by exactly the same argument we used in Lecture 19X, replacing $\mathrm{Stone}_{\mathcal{C}}$ by the subcategory $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{fr}}$. □

We begin with some preliminaries. Suppose we are given a collection of models $\{M_i\}_{i \in I}$ of $\mathcal{C}$, indexed by a set I. In Lecture 20X, we defined the ultraproduct $(\prod_{i \in I} M_i)/\mathcal{U}$ associated to an ultrafilter $\mathcal{U}$ on I: it is the functor from $\mathcal{C}$ to $\mathcal{S}\mathrm{et}$ given by the construction $C \mapsto (\prod_{i \in I} M_i(C))/\mathcal{U}$. In the special case where each M_i is equal to some fixed model M, we denote this ultraproduct by $M^I/\mathcal{U}$ and refer to it as the *ultrapower of M with respect to the ultrafilter* $\mathcal{U}$. Note that the diagonal embedding $M(C) \mapsto M(C)^I$ induces a map

$$M(C) \to M(C)^I \to \varinjlim_{J \in \mathcal{U}} M(C)^J = (M^I/\mathcal{U})(C).$$

This map depends functorially on C, and can therefore be regarded as a morphism of models $\delta_M : M \to M^I/\mathcal{U}$. We will deduce Theorem 3 from the following:

Theorem 5. *Let $f : M \to N$ be an elementary map between models of $\mathcal{C}$. Then there exists a set I, an ultrafilter $\mathcal{U}$ on I, and a map of models $g : N \to M^I/\mathcal{U}$ for which the composite map*

$$M \xrightarrow{f} N \xrightarrow{g} M^I/\mathcal{U}$$

coincides with the diagonal embedding δ_M.

Exercise 6. Let M be a model of $\mathcal{C}$, I a set, and $\mathcal{U}$ an ultrafilter on I. Show that the map of models $\delta_M : M \to M^I/\mathcal{U}$ is elementary.

Proof of Theorem 3 from Theorem 5. Suppose that $f : M \to N$ is an elementary map between models of $\mathcal{C}$, and that we are given a point $\eta \in \mathscr{F}(N)$ which belongs to the equalizer

$$\mathrm{Eq}(\mathscr{F}(N) \rightrightarrows \prod_{M \to N \rightrightarrows P} \mathscr{F}(P)).$$

We wish to show that η can be lifted uniquely to an element of $\mathscr{F}(M)$.

Choose $g : N \to M^I/\mathcal{U}$ as in Theorem 5, and let $\overline{\eta}$ denote the image of η in $\mathscr{F}(M^I/\mathcal{U})$. Since $\mathscr{F}$ commutes with ultraproducts (Remark 1), we can identify $\overline{\eta}$ with an element of $\mathscr{F}(M)^I/\mathcal{U}$. Choose a representative $\{\eta_i \in \mathscr{F}(M)\}_{i \in I}$ for $\overline{\eta}$. We have a commutative diagram of models

$$\begin{array}{ccc} M & \xrightarrow{\delta_M} & M^I/\mathcal{U} \\ \downarrow{\scriptstyle \delta_M} & & \downarrow{\scriptstyle (\delta_M)^I/\mathcal{U}} \\ M^I/\mathcal{U} & \xrightarrow{\delta_{M^I/\mathcal{U}}} & (M^I/\mathcal{U})^I/\mathcal{U}. \end{array}$$

Consequently, our hypothesis on η guarantees that the maps

$$\delta_M^I/\mathcal{U}, \delta_{M^I/\mathcal{U}} : M^I/\mathcal{U} \to (M^I/\mathcal{U})^I/\mathcal{U}$$

carry $\overline{\eta}$ to the same element of

$$\mathscr{F}((M^I/\mathcal{U})^I/\mathcal{U}) \simeq (\mathscr{F}(M)^I/\mathcal{U})^I/\mathcal{U}.$$

Unwinding the definitions, this tells us that the set $\{i \in I : \{j \in I : \eta_i = \eta_j\} \in \mathcal{U}\}$ belongs to $\mathcal{U}$. In particular, it is nonempty: that is, there exists some $i \in I$ such that $\eta_i = \eta_j$ for almost all $j \in I$ (with respect to the ultrafilter $\mathcal{U}$). We will complete the proof by showing that η is the image of $\eta_i \in \mathscr{F}(M)$. To prove this, we use the commutativity of the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow{\scriptstyle \delta_M} & & \downarrow{\scriptstyle \delta_N} \\ M^I/\mathcal{U} & \xrightarrow{f^I/\mathcal{U}} & N^I/\mathcal{U} \end{array}$$

to observe that the maps $(f^I/\mathcal{U}) \circ g, \delta_N : N \to N^I/\mathcal{U}$ agree on M, and therefore carry η to the same element of $N^I/\mathcal{U}$. It follows that the set $\{j \in J : \mathscr{F}(f)(\eta_j) = \eta\}$ belongs to $\mathcal{U}$, and therefore has nonempty intersection with the set $\{j \in J : \eta_j = \eta_i\}$. $\square$

Proof of Theorem 5. To avoid confusion, let us use the notation T_M to denote the image of a model $M \in \mathrm{Mod}(\mathcal{C})$ under the inclusion $\mathrm{Mod}(\mathcal{C})^{\mathrm{op}} \hookrightarrow \mathrm{Pro}(\mathcal{C})$. The elementary map $f : M \to N$ can then be identified with a map of pro-objects $T_N \to T_M$, which we will denote by T_f. Our assumption that f is elementary guarantees that T_f is an effective epimorphism in $\mathrm{Pro}(\mathcal{C})$, and can therefore be realized as the limit of an inverse system $\{u_\alpha : C_\alpha \to D_\alpha\}$, where each u_α is an effective epimorphism in $\mathcal{C}$. Without loss of generality, we may assume that this inverse limit is indexed by the opposite of partially ordered set I which is *directed* (so that every finite subset of I has an upper bound in I).

For each $\alpha \in I$, set $P_\alpha = C_\alpha \times_{D_\alpha} T_M$ (where the fiber product is formed in $\mathrm{Pro}(\mathcal{C})$. Since M is a model, each of the maps $T_M \to D_\alpha$ factors through u_α. A choice of factorization determines a section $s_\alpha : T_M \to P_\alpha$ of the projection map $P_\alpha \to T_M$. To avoid confusion, let us write F_α for the image of P_α in the opposite category $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{S}\mathrm{et}) \simeq \mathrm{Pro}(\mathcal{C})^{\mathrm{op}}$, so that each s_α can be viewed as a natural transformation of functors $F_\alpha \to M$. Note that we have $T_N \simeq \varprojlim_\alpha P_\alpha$ in $\mathrm{Pro}(\mathcal{C})$, so that $N \simeq \varinjlim_\alpha F_\alpha$ in $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{S}\mathrm{et})$.

Let $\mathcal{U}_0$ be the collection of all subsets $J \subseteq I$ for which there exists some $\alpha \in I$ such that $\{\beta \in I : \beta \geq \alpha\} \subseteq J$. Our assumption that I is directed guarantees that $\mathcal{U}_0$ is a (nontrivial) filter on I. We can therefore choose an ultrafilter $\mathcal{U}$ which contains $\mathcal{U}_0$. For each object $C \in \mathcal{C}$ and each $\alpha \in I$, we have a canonical map $F_\alpha(C) \to \prod_{\beta \geq \alpha} F_\beta(C)$ given by the transition maps in the direct system $\{F_\beta(C)\}_{\beta \in I}$, which induces a map from $F_\alpha(C)$ to the ultraproduct $(\prod_{\beta \in I} F_\beta(C))/\mathcal{U}$. This construction depends functorially on α, and therefore yields a map

$$N(C) = \varinjlim_{\alpha \in I} F_\alpha(C) \to \varinjlim_{\alpha \in I} (\prod_{\beta \geq \alpha} F_\beta(C)) \to \varinjlim_{J \in \mathcal{U}} \prod_{\beta \in J} F_\beta(C) = (\prod_{\beta \in I} F_\beta(C))/\mathcal{U}.$$

Composing with the map

$$(\prod_{\beta \in I} F_\beta(C))/\mathcal{U} \xrightarrow{\{s_\beta\}} (\prod_{\beta \in I} M(C))/\mathcal{U} = (M^I/\mathcal{U})(C),$$

we obtain a map $N(C) \to (M^I/\mathcal{U})(C)$ which depends functorially on C, and can therefore be regarded as a map of models $N \to M^I/\mathcal{U}$. Our assumption that each s_α is a section of the projection map guarantees the the composition $M \to N \to M^I/\mathcal{U}$ agrees with the diagonal embedding. $\square$