Lecture 20: $\mathcal{X}$-Locales

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Let $f : \mathcal{Y} \to \mathcal{X}$ be a geometric morphism of topoi. In the previous lecture, we proved that if f is localic, then it can be recovered from any of the following three interchangeable pieces of data:

(a) The object $\Omega_{\mathcal{Y}/\mathcal{X}} \in \mathcal{X}$, together with its partial order $\Omega^{\subseteq}_{\mathcal{Y}/\mathcal{X}} \subseteq \Omega_{\mathcal{Y}/\mathcal{X}} \times \Omega_{\mathcal{Y}/\mathcal{X}}$.

(b) The functor $\mathcal{X}^{\mathrm{op}} \to \{\text{Posets}\}$ given by $X \mapsto \mathrm{Sub}(f^*X)$.

(c) The category $\mathrm{Loc}(f)$ obtained by applying the Grothendieck construction to the functor $X \mapsto \mathrm{Sub}(f^*X)$.

In this lecture, it will be convenient to adopt perspective (b). Our goal is to address the following:

Question 1. Let $\mathcal{L} : \mathcal{X}^{\mathrm{op}} \to \{\text{Posets}\}$ be a functor. Under what conditions does there exist a geometric morphism $f : \mathcal{Y} \to \mathcal{X}$ and a natural isomorphism $\mathcal{L} \simeq \mathrm{Sub}(f^*(\bullet))$?

We begin by recording some necessary conditions.

Proposition 2. *Let $f : \mathcal{Y} \to \mathcal{X}$ be a geometric morphism of topoi and define $\mathcal{L} : \mathcal{X}^{\mathrm{op}} \to \{Posets\}$ by the formula $\mathcal{L}(X) = \mathrm{Sub}(f^*(X))$. Then:*

(1) *The functor $\mathcal{L} : \mathcal{X}^{\mathrm{op}} \to \{Posets\}$ is a sheaf (with respect to the canonical topology on $\mathcal{X}$).*

(2) *For each $X \in \mathcal{X}$, the poset $\mathcal{L}(X)$ is a locale.*

(3) *For each morphism $g : X' \to X$ in $\mathcal{X}$, let $g^* = \mathcal{L}(g)$ denote the associated map of posets from $\mathcal{L}(X)$ to $\mathcal{L}(X')$. Then g^* determines an open morphism of locales $\mathcal{L}(X') \to \mathcal{L}(X)$. In particular, it has a left adjoint $g_! : \mathcal{L}(X') \to \mathcal{L}(X)$.*

(4) *For every pullback diagram*

$$\begin{array}{ccc} U' & \xrightarrow{g'} & U \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow{g} & X, \end{array}$$

we have an equality $h'_! \circ g'^* = g^* \circ h_!$ of maps from $\mathcal{L}(U)$ to $\mathcal{L}(X')$.

Proof. Assertions (1), (2), and (3) were noted in the previous lecture. To prove (4), let $V \in \mathcal{L}(U) = \mathrm{Sub}(f^*U)$. Note that we have a pullback diagram

$$\begin{array}{ccc} f^*U' & \longrightarrow & f^*U \\ \downarrow & & \downarrow \\ f^*X' & \longrightarrow & f^*X. \end{array}$$

Then $h_!V$ is the image of the composite map $V \subseteq f^*U \to f^*X$, and $h'_! g'^*(V)$ is the image of the map

$$V \times_{f^*U} f^*U' \simeq V \times_{f^*X} f^*X' \to f^*X'.$$

The equality $(g^* \circ h_!(V) = (h'_! \circ g'^*)(V)$ now follows from the fact that the formation of images in the topos $\mathcal{Y}$ is compatible with pullback. $\square$

Definition 3. Let $\mathcal{X}$ be a topos. An $\mathcal{X}$-*locale* is a functor $\mathcal{L} : \mathcal{X}^{\mathrm{op}} \to \{\text{Posets}\}$ which satisfies conditions (1) through (4) of Proposition 2.

Remark 4. We will refer to condition (4) of Proposition 2 as the *Beck-Chevalley condition* for the functor $\mathcal{L}$. Note that for *any* commutative diagram σ :

$$\begin{array}{ccc} U' & \xrightarrow{g'} & U \\ \big\downarrow{\scriptstyle h'} & & \big\downarrow{\scriptstyle h} \\ X' & \xrightarrow{g} & X, \end{array}$$

in the topos $\mathcal{X}$, we automatically have an inequality $h'_! \circ g'^* \leq g^* \circ h_!$. The Beck-Chevalley condition asserts that the reverse inequality holds when σ is a pullback square.

Remark 5. Conditions (1), (2), and (3) of Proposition 2 can be summarized by saying that the functor $\mathcal{L} : \mathcal{X}^{\mathrm{op}} \to \{\text{Posets}\}$ can be regarded as a sheaf on $\mathcal{X}$ taking values in the opposite of the category of locales and open morphisms.

We can restate Proposition 2 as follows: for every geometric morphism $f : \mathcal{Y} \to \mathcal{X}$, the construction $X \mapsto \mathrm{Sub}(f^*X)$ determines an $\mathcal{X}$-locale. We now prove the converse:

Proposition 6. *Let $\mathcal{X}$ be a topos and let $\mathcal{L} : \mathcal{X}^{\mathrm{op}} \to \{\text{Posets}\}$ be an $\mathcal{X}$-locale. Then there exists a localic geometric morphism $f : \mathcal{Y} \to \mathcal{X}$ and a natural isomorphism $\mathcal{L}(X) \simeq \mathrm{Sub}(f^*X)$.*

Remark 7. In Lecture 19, we saw that a localic geometric morphism $f : \mathcal{Y} \to \mathcal{X}$ can be recovered from the associated $\mathcal{X}$-locale. Combining this with Proposition 6, we obtain a dictionary

$$\{\text{Topoi localic over } \mathcal{X}\} \simeq \{\mathcal{X}\text{-locales}\}.$$

We will later refine this picture to an equivalence of categories.

Proof of Proposition 6. Lecture 19 tells us what we need to do. Let $\widetilde{\mathcal{X}}$ denote the category obtained by applying the Grothendieck construction to the functor $\mathcal{L}$. More concretely, it can be described as follows:

- The objects of $\widetilde{\mathcal{X}}$ are pairs (X, U), where $X \in \mathcal{X}$ and $U \in \mathcal{L}(X)$.
- A morphism from (X', U') to (X, U) is a morphism $g : X' \to X$ in the topos $\mathcal{X}$ satisfying $U' \leq g^*(U)$ (or equivalently $g_!(U') \leq U$).

Let us say that a collection of maps $\{g_i : (X_i, U_i) \to (X, U)\}_{i \in I}$ in $\widetilde{\mathcal{X}}$ is a *covering* if $U = \bigvee g_{i!} U_i$ in $\mathcal{L}(X)$. We claim that this determines a Grothendieck topology on $\widetilde{\mathcal{X}}$. We check that the collection of coverings is stable under pullback (the remaining axioms for a Grothendieck topology are easy and left as an exercise).

Suppose we are given a covering $\{g_i : (X_i, U_i) \to (X, U)$ and an arbitrary morphism $h : (Y, V) \to (X, U)$. For each $i \in I$, we can form the fiber product $(X_i, U_i) \times_{(X,U)} (Y, V)$ in the category $\widetilde{X}$; it is given by $(X_i \times_X Y, \pi_i^* U_i \wedge \pi_i'^* V)$, where $\pi_i : U_i \times_X Y \to U_i$ and $\pi'_i : U_i \times_X Y \to Y$ are the two projection maps. We claim that the family of maps $\{\pi'_i : (X_i \times_X Y, \pi_i^* U_i \wedge \pi_i'^* V) \rightarrow (Y, V)\}_{i \in I}$ is also a covering. For this, we compute

$$\begin{aligned}
\bigvee_{i \in I} \pi'_{i!}(\pi_i^* U_i \wedge \pi_i'^* V) &= \bigvee_{i \in I} (\pi'_{i!}(\pi_i^* U_i) \wedge V) \\
&= (\bigvee_{i \in I} \pi'_{i!}(\pi_i^* U_i)) \wedge V \\
&= (\bigvee_{i \in I} h^* g_{i!} U_i) \wedge V \\
&= h^*(\bigvee_{i \in I} g_{i!} U_i) \wedge V \\
&= h^*(U) \wedge V \\
&= V.
\end{aligned}$$

Here the first equality follows from the projection formula for the open morphism of locales $\mathcal{L}(U_i \times_X Y) \rightarrow \mathcal{L}(Y)$, the second equality from the distributive law in the locale $\mathcal{L}(Y)$, the third from the Beck-Chevalley property for $\mathcal{L}$, the fourth from the fact that the map h^* preserves joins, the fifth from the assumption that $\{(X_i, U_i) \rightarrow (X, U)\}_{i \in I}$ is a covering, and the sixth from the inequality $V \subseteq h^* U$.

We now encounter a bit of a technical annoyance: the category $\widetilde{\mathcal{X}}$ is not small, so it is not *a priori* clear that the category $\mathrm{Shv}(\widetilde{\mathcal{X}})$ is a topos. To address this, let us choose a small full subcategory $\mathcal{X}_0 \subseteq \mathcal{X}$ which generates $\mathcal{X}$ and is closed under finite limits, and set $\widetilde{\mathcal{X}}_0 = \widetilde{\mathcal{X}} \times_{\mathcal{X}} \mathcal{X}_0$. Then $\widetilde{\mathcal{X}}_0$ inherits a Grothendieck topology (with the notion of covering defined as above), so that $\mathrm{Shv}(\widetilde{\mathcal{X}}_0)$ is a topos. Moreover, the inclusion functor $\mathrm{Shv}(\widetilde{\mathcal{X}}_0) \hookrightarrow \mathrm{Fun}(\widetilde{\mathcal{X}}_0^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ admits a left adjoint (given by sheafification), which we will denote by L. (One can show that the restriction map $\mathrm{Shv}(\widetilde{\mathcal{X}}) \rightarrow \mathrm{Shv}(\widetilde{\mathcal{X}}_0)$ is an equivalence of categories, but we will not need this). Let $\widetilde{h} : \widetilde{\mathcal{X}}_0 \rightarrow \mathrm{Shv}(\widetilde{\mathcal{X}}_0)$ denote the sheafified Yoneda embedding $\widetilde{h}(X, U) = L h_{(X,U)}$ (beware that the topology on $\widetilde{\mathcal{X}}_0$ is not subcanonical, so the sheafification is necessary). Since L and the Yoneda embedding preserve finite limits, the functor $\widetilde{h}$ also preserves finite limits.

For each object $X \in \mathcal{X}$, let $\mathbf{1}_X$ denote the largest element of $\mathcal{L}(X)$. Then the construction $X \mapsto (X, \mathbf{1}_X)$ determines a functor $\mathcal{X} \rightarrow \widetilde{\mathcal{X}}$ which preserves finite limits. We claim that it also preserves coverings: that is, for every covering $\{g_i : X_i \rightarrow X\}$ in $\mathcal{X}$, the family $\{(X_i, \mathbf{1}_{X_i}) \rightarrow (X, \mathbf{1}_X)\}$ is a covering in $\mathcal{X}$. To prove this, we must show that if $U \in \mathcal{L}(X)$ has the property that $\mathbf{1}_{X_i} \leq g_i^* U$ for each $i \in I$, then $U = \mathbf{1}_X$. This follows from our assumption that $\mathcal{L}$ is a sheaf (since $g_i^* U = \mathbf{1}_{X_i}$ for each i). It follows that there is an essentially unique geometric morphism of topoi $f' : \mathrm{Shv}(\widetilde{\mathcal{X}}_0) \rightarrow \mathrm{Shv}(\mathcal{X}_0)$ satisfying $f'^* h_X \simeq \widetilde{h}(X, U)$ for each $X \in \mathcal{X}_0$. Composing with the equivalence $\mathcal{X} \simeq \mathrm{Shv}(\mathcal{X}_0)$ given by the Yoneda embedding, we obtain a geometric morphism $f : \mathrm{Shv}(\widetilde{\mathcal{X}}_0) \rightarrow \mathcal{X}$ with the property that $f^* X \simeq \widetilde{h}(X, \mathbf{1}_X)$ for each $X \in \mathcal{X}_0$. We will complete the proof by showing that this functor has the desired properties.

We first note that the topos $\mathrm{Shv}(\widetilde{\mathcal{X}}_0)$ is generated by objects of the form $\widetilde{h}(X, U)$, where $X \in \mathcal{X}_0$ and $U \in \mathcal{L}(X)$. For every such object, we have a monomorphism $(X, U) \rightarrow (X, \mathbf{1}_X)$ in the category $\widetilde{\mathcal{X}}_0$, hence also a homomorphism $\widetilde{h}(X, U) \rightarrow \widetilde{h}(X, \mathbf{1}_X) \simeq f^* X$ in the category $\mathrm{Shv}(\widetilde{\mathcal{X}}_0)$. It follows that the geometric morphism f is localic.

To complete the proof, it will suffice to construct a natural isomorphism $\mathcal{L} \simeq \mathrm{Sub}(f^*(\bullet))$ of functors $\mathcal{X}^{\mathrm{op}} \rightarrow \{\mathrm{Posets}\}$. For each $X \in \mathcal{X}_0$, we have a map

$$\rho_X : \mathcal{L}(X) \rightarrow \mathrm{Sub}(f^* X) \simeq \mathrm{Sub}(\widetilde{h}(X, \mathbf{1}_X)),$$

given by $\rho_X(U) = \widetilde{h}(X, U)$. Note that ρ_X depends functorially on X. We will show that each ρ_X is bijective, so that $\{\rho_X\}_{X \in \mathcal{X}_0}$ is a natural isomorphism of functors $\mathcal{L} \,|_{\mathcal{X}_0^{\mathrm{op}}} \simeq \mathrm{Sub}(f^*(\bullet))|_{\mathcal{X}_0^{\mathrm{op}}}$. Since both sides are sheaves with respect to the canonical topology on $\mathcal{X}$, it follows that this natural isomorphism extends uniquely to an isomorphism of functors $\mathcal{L} \simeq \mathrm{Sub}(f^*(\bullet))$.

Let us henceforth regard $X \in \mathcal{X}_0$ as fixed. We will construct an inverse to the map ρ_X. Let $\mathscr{F}$ be a subsheaf of $f^* X = L h_{(X, \mathbf{1}_X)}$. For each $U \in \mathcal{L}(X)$, let $\iota_U : (X, U) \to (X, \mathbf{1}_X)$ denote the morphism in $\widetilde{\mathcal{X}}_0$ given by the identity map from X to itself, which we can regard as an element of $h_{(X, \mathbf{1}_X)}(X, U)$. Let $\widetilde{\iota}_U$ denote the image of ι_U as a section of the associated sheaf $L h_{(X, \mathbf{1}_X)} = f^* X$. Consider the set

$$\mathcal{U} = \{U \in \mathcal{L}(X) : \widetilde{\iota}_U \in \mathscr{F}(X, U) \subseteq (f^* X)(X, U)\}.$$

Note that $\mathcal{U}$ is closed under joins (this follows by applying the sheaf condition to $\mathscr{F}$, since the family $\{(X, U_i) \to (X, \bigvee U_i)\}$ is a covering for any family of elements $\{U_i \in \mathcal{L}(X)\}_{i \in i}$). It follows that $\mathcal{U}$ contains a largest element U. The construction $\mathscr{F} \mapsto U$ determines a map of posets $\psi : \mathrm{Sub}(f^* X) \to \mathcal{L}(X)$.

We first claim that the composition $\mathcal{L}(X) \xrightarrow{\rho_X} \mathrm{Sub}(f^* X) \xrightarrow{\psi} \mathcal{L}(X)$ is the identity. Fix an object $U \in \mathcal{L}(X)$. Then ι_U belongs to the sub-presheaf $h_{(X,U)} \subseteq h_{(X,\mathbf{1})}$ (when evaluated at (X, U)), so that $\widetilde{\iota}_U$ belongs to the subsheaf $\widetilde{h}(X, U) \subseteq \widetilde{h}(X, \mathbf{1}_X)$ (when evaluated at (X, U)). We wish to show that U is the largest element of $\mathcal{L}(X)$ with this property. Let V be any element of $\mathcal{L}(X)$, and assume that $\widetilde{\iota}_V$ belongs to the subsheaf $\widetilde{h}(X, U) \subseteq \widetilde{h}(X, \mathbf{1}_X)$ (when evaluated at (X, V)). Then we can choose a covering $\{g_i : (X_i, V_i) \to (X, V)\}$ in $\mathcal{X}_0$ with the property that each of the composite maps $(X_i, V_i) \to (X, V) \subseteq (X, \mathbf{1}_X)$ belongs to $h_{(X,U)}(X_i, V_i)$. It follows that we have $g_{i!} V_i \leq U$ for each i, so that $V = \bigvee g_{i!} V_i \leq U$. This completes the proof that $\psi \circ \rho_X$ is the identity.

We now prove that the composition $\rho_X \circ \psi : \mathrm{Sub}(f^* X) \to \mathrm{Sub}(f^* X)$ is the identity. Let $\mathscr{F}$ be a subsheaf of $f^* X$ and form a pullback diagram of presheaves

$$\begin{array}{ccc} \mathscr{F}_0 & \longrightarrow & \mathscr{F} \\ \downarrow & & \downarrow \\ h_{(X, \mathbf{1}_X)} & \longrightarrow & f^* X. \end{array}$$

Set $U = \psi(\mathscr{F})$. By construction, U is the largest element of $\mathcal{U}(X)$ for which $\mathscr{F}_0$ contains $h_{(X,U)}$ (as subfunctors of $h_{(X, \mathbf{1}_X)}$. We will complete the proof by showing that $\mathscr{F}_0 = h_{(X,U)}$; it then follows by the left exactness of sheafification that $\mathscr{F} = L \mathscr{F}_0 = L h_{(X,U)} = \rho_X(U)$. Fix an object $(Y, V) \in \mathcal{X}_0$ and a point η of $\mathscr{F}_0(Y, V)$, which we can identify with a map $g : Y \to X$. The map $(Y, V) \to (X, g_! V)$ is a covering in $\mathcal{X}_0$. Since membership in the subsheaf $\mathscr{F} \subseteq f^* X$ can be tested locally, we conclude that $\widetilde{\iota}_{g_! V}$ belongs to $\mathscr{F}(X, g_! V)$, so that $g_! V \subseteq U$. It follows that η belongs to the sub-pre-sheaf $h_{(X,U)} \subseteq \mathscr{F}_0$, as desired. $\square$