Lecture 19X-Sheaves on $\mathrm{Stone}_{\mathcal{C}}$

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Let $\mathcal{C}$ be an essentially small pretopos, which we regard as fixed throughout this lecture. In Lecture 15X, we constructed a fully faithful embedding

$$\mathrm{Shv}(\mathcal{C}) \hookrightarrow \mathrm{Shv}(\mathrm{Pro}(\mathcal{C})) \simeq \mathrm{Shv}(\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})) \simeq \mathrm{Shv}(\mathrm{Stone}_{\mathcal{C}}) \subseteq \mathrm{Fun}(\mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}}, \mathcal{S}\mathrm{et}).$$

The essential image of this embedding consists of those functors $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ which commute with filtered colimits and are sheaves with respect to the Grothendieck topology of Lecture 18X. Any such functor must satisfy the following condition:

(a) The functor $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ preserves finite products: that is, it carries finite coproducts in $\mathrm{Stone}_{\mathcal{C}}$ to finite products in the category of sets.

In Lecture 17X, we proved that if $\mathscr{F}$ is a functor satisfying (a), then it commutes with filtered colimits if and only if it satisfies the following additional conditions:

(b) For every object $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{C}}$ and every point $x \in X$, the canonical map

$$\varinjlim_{x \in U} \mathscr{F}(U, \mathcal{O}_X |_U) \to \mathscr{F}(\{x\}, \mathcal{O}_{X,x})$$

is bijective; here the colimit is taken over all clopen neighborhoods $U \subseteq X$ of the point x.

(c) The composite functor

$$\mathrm{Mod}(\mathcal{C}) \hookrightarrow \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \xrightarrow{\mathscr{F}} \mathcal{S}\mathrm{et}$$

commutes with filtered colimits.

Our goal in this section is to characterize those functors $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ which are sheaves. It is easy to see that if $\mathscr{F}$ is a sheaf, then it must satisfy condition (a) above. We will prove the following partial converse:

Theorem 1. *Let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a functor satisfying conditions (a) and (b). Then $\mathscr{F}$ is a sheaf if and only if it satisfies the following further condition:*

(d) *For every elementary morphism $f : M \to N$ in $\mathrm{Mod}(\mathcal{C})$, we have an equalizer diagram*

$$\mathscr{F}(M) \to \mathscr{F}(N) \rightrightarrows \prod \mathscr{F}(P)$$

where the product is taken over all commutative diagrams

$$M \xrightarrow{f} N \rightrightarrows P$$

in $\mathrm{Mod}(\mathcal{C})$.

Here we identify $\mathrm{Mod}(\mathcal{C})$ *with the full subcategory of* $\mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}}$ *spanned by those pairs* $(X, \mathcal{O}_X)$*, where* X *is a singleton.*

Warning 2. In the formulation of condition (d), the product $\prod \mathscr{F}(P)$ is an ill-defined object, because it is indexed by a proper class. However, the equalizer of the diagram $\mathscr{F}(N) \rightrightarrows \prod \mathscr{F}(P)$ is still well-defined as a subset of $\mathscr{F}(N)$.

Remark 3. Condition (d) is equivalent to the requirement that the restriction $\mathscr{F}|_{\mathrm{Mod}(\mathcal{C})}$ is a sheaf on the category $\mathrm{Mod}(\mathcal{C})^{\mathrm{op}}$, where we consider a collection of morphisms $\{M \to N_i\}_{i \in I}$ in $\mathrm{Mod}(\mathcal{C})$ to be a covering in $\mathrm{Mod}(\mathcal{C})^{\mathrm{op}}$ if at least one of the maps $M \to N_i$ is elementary (it follows from the amalgamation property of the previous lecture that this notion of covering defines a Grothendieck topology on $\mathrm{Mod}(\mathcal{C})^{\mathrm{op}}$).

Corollary 4. *The topos* $\mathrm{Shv}(\mathcal{C})$ *can be identified with the full subcategory of* $\mathrm{Fun}(\mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ *spanned by those functors* $\mathscr{F}$ *which satisfy conditions* (a)*,* (b)*,* (c) *and* (d) *above.*

Before giving the proof of Theorem 1, it will be convenient to revisit a construction from Lecture 17X, which gives a convenient reformulation of conditions (a) and (b).

Notation 5. Let $(X, \mathcal{O}_X)$ be an object of $\mathrm{Stone}_{\mathcal{C}}$, and let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a functor. Let $\mathcal{U}_0(X)$ denote the collection of all clopen subsets of X. We define a functor

$$\mathscr{F}(\mathcal{O}_X) : \mathcal{U}_0(X)^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$$

by the formula

$$\mathscr{F}(\mathcal{O}_X)(U) = \mathscr{F}(U, \mathcal{O}_X|_U).$$

Note that the functor $\mathscr{F}$ satisfies condition (a) above if and only if, for every object $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{C}}$, the functor $\mathscr{F}(\mathcal{O}_X) : \mathcal{U}_0(X)^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ carries disjoint unions in $\mathcal{U}_0(X)$ to products in $\mathcal{S}\mathrm{et}$. In this case, $\mathscr{F}(\mathcal{O}_X)$ extends uniquely to a sheaf of sets on X, which we will also denote by $\mathscr{F}(\mathcal{O}_X)$.

By construction, the stalk of $\mathscr{F}(\mathcal{O}_X)$ at a point $x \in X$ is given by the direct limit $\varinjlim_{x \in U} \mathscr{F}(U, \mathcal{O}_X|_U)$. We therefore have a canonical map $\mathscr{F}(\mathcal{O}_X)_x \to \mathscr{F}(\mathcal{O}_{X,x})$ (here we abuse notation by identifying the model $\mathcal{O}_{X,x}$ with the object $(\{x\}, \mathcal{O}_{X,x}) \in \mathrm{Stone}_{\mathcal{C}}$). Condition (b) can then be restated as follows:

(b') For each $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{C}}$ and each point $x \in X$, the map $\mathscr{F}(\mathcal{O}_X)_x \to \mathscr{F}(\mathcal{O}_{X,x})$ is a bijection.

Remark 6. Let $(Y, \mathcal{O}_Y)$ be an object of $\mathrm{Stone}_{\mathcal{C}}$ and suppose we are given a map of Stone spaces $f : X \to Y$, so that $f^* \mathcal{O}_Y$ is an X-model of $\mathcal{C}$. If $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ is a functor satisfying condition (a), then Notation 5 determines set-valued sheaves

$$\mathscr{F}(\mathcal{O}_Y) \in \mathrm{Shv}(Y) \qquad \mathscr{F}(f^* \mathcal{O}_Y) \in \mathrm{Shv}(X)$$

together with a comparison map $f^* \mathscr{F}(\mathcal{O}_Y) \to \mathscr{F}(f^* \mathcal{O}_Y)$ in $\mathrm{Shv}(X)$. If $\mathscr{F}$ satisfies condition (b), then this comparison map is an isomorphism of sheaves on X (this can be checked on stalks, where it follows from (b')).

Remark 7. Let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a functor satisfying (a) and (b). Then, for each $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{C}}$, the canonical map

$$\mathscr{F}(X, \mathcal{O}_X) \to \prod_{x \in X} \mathscr{F}(\{x\}, \mathcal{O}_{X,x})$$

is injective. This follows from the fact that a section s of the sheaf $\mathscr{F}(\mathcal{O}_X) \in \mathrm{Shv}(X)$ is determined by its stalks $\{s_x\}_{x \in X}$.

We can now prove the "easy" direction of Theorem 1. Let

$$\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$$

be a sheaf (so that is satisfies condition (a)), and suppose that $\mathscr{F}$ also satisfies condition (b). We wish to show that it satisfies condition (d). Suppose we are given an elementary morphism $M \to N$ in $\mathrm{Mod}(\mathcal{C})$. Then the induced map

$$(*, N) \to (*, M)$$

is a covering in $\mathrm{Stone}_{\mathcal{C}}$ (see Lecture 18X). It follows that the canonical map $\mathscr{F}(M) \to \mathscr{F}(N)$ is injective, and that its image consists of those elements $s \in \mathscr{F}(N)$ which satisfy the following condition:

- $(*)$ For every object $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{C}}$ equipped with a pair of maps $(X, \mathcal{O}_X) \rightrightarrows (*, N)$ which are coequalized by $(*, N) \to (*, M)$, the element s belongs to the equalizer $\mathrm{Eq}(\mathscr{F}(N) \rightrightarrows \mathscr{F}(X, \mathcal{O}_X))$.

To verify (d), we must show that it suffices to check the criterion of $(*)$ in the case where X is a single point. This follows from the injectivity of the map $\mathscr{F}(X, \mathcal{O}_X) \to \prod_{x \in X} \mathscr{F}(\{x\}, \mathcal{O}_{X,x})$ (Remark 7).

We now tackle the hard direction. Assume that $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ satisfies conditions (a), (b), and (d); we wish to show that $\mathscr{F}$ is a sheaf. Choose a covering $\{(X_i, \mathcal{O}_{X_i}) \to (X, \mathcal{O}_X)\}_{i \in I}$ in the category $\mathrm{Stone}_{\mathcal{C}}$. For every pair $i, j \in I$, we can identify $(X_i, \mathcal{O}_{X_i})$, $(X_j, \mathcal{O}_{X_j})$, and $(X, \mathcal{O}_X)$ with weakly projective pro-objects $\Gamma(X_i; \mathcal{O}_{X_i})$, $\Gamma(X_j; \mathcal{O}_{X_j})$, and $\Gamma(X; \mathcal{O}_X)$. We can then form the fiber product

$$\Gamma(X_i; \mathcal{O}_{X_i}) \times_{\Gamma(X; \mathcal{O}_X)} \Gamma(X_j; \mathcal{O}_{X_j})$$

in $\mathrm{Pro}(\mathcal{C})$. This fiber product might not be weakly projective. However, it we can choose a *covering* by a weakly projective pro-object, which we can then write in the form $\Gamma(X_{ij}, \mathcal{O}_{X_{ij}})$ for some $(X_{ij}, \mathcal{O}_{X_{ij}}) \in \mathrm{Stone}_{\mathcal{C}}$). In order to show that $\mathscr{F}$ is a sheaf, we must verify that the diagram

$$\mathscr{F}(X, \mathcal{O}_X) \to \prod_i \mathscr{F}(X_i, \mathcal{O}_{X_i}) \rightrightarrows \prod_{i,j} \mathscr{F}(X_{ij}, \mathcal{O}_{X_{ij}})$$

is an equalizer diagram in the category of sets. Since every covering admits a finite subcover, it suffices to check this in the case where the set I is finite. In this case, we can form the coproducts

$$(Y, \mathcal{O}_Y) = \coprod_{i \in I} (X_i, \mathcal{O}_{X_i}) \qquad (Z, \mathcal{O}_Z) = \coprod_{i,j \in I} (X_{ij}, \mathcal{O}_{X_{ij}}).$$

Using condition (a), we are reduced to showing that the diagram

$$\mathscr{F}(X, \mathcal{O}_X) \to \mathscr{F}(Y, \mathcal{O}_Y) \rightrightarrows \mathscr{F}(Z, \mathcal{O}_Z)$$

is an equalizer diagram of sets.

Let $\mathscr{G}$ denote the direct image of $\mathscr{F}(\mathcal{O}_Y) \in \mathrm{Shv}(Y)$ along the projection map $Y \to X$, and let $\mathscr{H}$ denote the direct image of $\mathscr{F}(\mathcal{O}_Z)$ along the projection $Z \to X$. We then have a commutative diagram

$$\mathscr{F}(\mathcal{O}_X) \to \mathscr{G} \rightrightarrows \mathscr{H}$$

in $\mathrm{Shv}(X)$, and we wish to show that it becomes an equalizer diagram in $\mathcal{S}\mathrm{et}$ after taking global sections. To prove this, it will suffice to show that the above diagram is an equalizer in $\mathrm{Shv}(X)$. This can be checked stalkwise: that is, we are reduced to showing that the map

$$\mathscr{F}(\mathcal{O}_X)_x \to \mathscr{G}_x \rightrightarrows \mathscr{H}_x$$

is an equalizer diagram of sets, for each point $x \in X$. Let $Y_x \subseteq Y$ and $Z_x \subseteq Z$ denote the inverse images of x. Using (b) (in the form of Remark 7), we are reduced to showing that the diagram

$$\mathscr{F}(\{x\}, \mathcal{O}_{X,x}) \xrightarrow{\psi} \mathscr{F}(Y_x, \mathcal{O}_Y|_{Y_x}) \rightrightarrows \mathscr{F}(Z_x, \mathcal{O}_Z|_{Z_x})$$

is an equalizer diagram of sets.

Since the map $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a covering, Lecture 18X shows that we an choose a point $y \in Y_x$ for which the map of stalks $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is elementary. Using condition (d), we deduce that the composite map

$$\mathscr{F}(\{x\}, \mathcal{O}_{X,x}) \xrightarrow{\psi} \mathscr{F}(Y_x, \mathcal{O}_Y|_{Y_x}) \to \mathscr{F}(\{y\}, \mathcal{O}_{Y,y})$$

is injective, so that ψ is injective. We will complete the proof by showing that every element s of the equalizer

$$\mathrm{Eq}(\mathscr{F}(Y_x, \mathcal{O}_Y|_{Y_x}) \rightrightarrows \mathscr{F}(Z_x, \mathcal{O}_Z|_{Z_x}))$$

belongs the image of ψ. Let $s_y \in \mathscr{F}(\{y\}, \mathcal{O}_{Y,y})$ denote the stalk of s at the point y. We first claim that s_y belongs to the image of the composite map

$$\mathscr{F}(\{x\}, \mathcal{O}_{X,x}) \xrightarrow{\psi} \mathscr{F}(Y_x, \mathcal{O}_Y|_{Y_x}) \to \mathscr{F}(\{y\}, \mathcal{O}_{Y,y}).$$

By virtue of (d), it will suffice to prove the following:

$(*')$ Given a model $P \in \mathrm{Mod}(\mathcal{C})$ and a commutative diagram of models

$$\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y} \rightrightarrows P,$$

the stalk s_y belongs to the equalizer $\mathrm{Eq}(\mathscr{F}(\mathcal{O}_{Y,y}) \rightrightarrows \mathscr{F}(P))$.

Let us identify P with an object of $\mathrm{Pro}(\mathcal{C})$, and form a pullback diagram

$$\begin{array}{ccc} \widetilde{P} & \longrightarrow & \Gamma(Z; \mathcal{O}_Z) \\ \downarrow & & \downarrow \\ P & \longrightarrow & \Gamma(Y; \mathcal{O}_Y) \times_{\Gamma(X; \mathcal{O}_X)} \Gamma(Y; \mathcal{O}_Y). \end{array}$$

in $\mathrm{Pro}(\mathcal{C})$. Here the right vertical map is an effective epimorphism, so the left vertical map is an effective epimorphism as well. The object $\widetilde{P}$ might not be weakly projective. However, we can choose an effective epimorphism $Q \to \widetilde{P}$, where Q is weakly projective. We can then write $Q = \Gamma(W, \mathcal{O}_W)$, for some object $(W, \mathcal{O}_W)$ in $\mathrm{Stone}_{\mathcal{C}}$. By construction, the map $(W, \mathcal{O}_W) \to (*, P)$ is a covering in $\mathrm{Stone}_{\mathcal{C}}$. Using Lecture 18X, we see that there exists a point $w \in W$ for which the map of models $P \to \mathcal{O}_{W,w}$ is elementary. Using condition (d), we see that the map $\mathscr{F}(P) \to \mathscr{F}(\mathcal{O}_{W,w})$ is injective. Consequently, to verify $(*')$, we are free to replace P by $\mathcal{O}_{W,w}$. Let $z \in Z$ denote the image of w under the map $(W, \mathcal{O}_W) \to (Z, \mathcal{O}_Z)$ in $\mathrm{Stone}_{\mathcal{C}}$, so that the diagram of $(*')$ refines to a diagram

$$\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y} \rightrightarrows \mathcal{O}_{Z,z} \to P.$$

We are therefore reduced to showing that s_y belongs to the equalizer $\mathrm{Eq}(\mathscr{F}(\mathcal{O}_{Y,y}) \rightrightarrows \mathscr{F}(\mathcal{O}_{Z,z}))$, which follows from our assumption that $s \in \mathrm{Eq}(\mathscr{F}(Y_x, \mathcal{O}_Y|_{Y_x}) \rightrightarrows \mathscr{F}(Z_x, \mathcal{O}_Z|_{Z_x}))$.

The above argument shows that we can write s_y as the image of an element $\overline{s} \in \mathscr{F}(\{x\}, \mathcal{O}_{X,x})$. We will complete the proof by showing that $\psi(\overline{s}) = s$. For this, it will suffice to show that $\psi(\overline{s})$ and s have the same image in the stalk $\mathscr{F}(\{y'\}, \mathcal{O}_{Y,y'})$ for each point y' in the fiber Y_x. Using the amalgamation property of the previous lecture, we see that there exists a commutative diagram of models σ :

$$\begin{array}{ccc} \mathcal{O}_{X,x} & \longrightarrow & \mathcal{O}_{Y,y} \\ \downarrow & & \downarrow \\ \mathcal{O}_{Y,y'} & \longrightarrow & N \end{array}$$

where the bottom horizontal map is elementary. As above, we can form a pullback diagram

$$\begin{array}{ccc} \widetilde{N} & \longrightarrow & \Gamma(Z; \mathcal{O}_Z) \\ \downarrow & & \downarrow \\ N & \longrightarrow & \Gamma(Y; \mathcal{O}_Y) \times_{\Gamma(X; \mathcal{O}_X)} \Gamma(Y; \mathcal{O}_Y) \end{array}$$

in $\mathrm{Pro}(\mathcal{C})$ and choose an effective epimorphism $\Gamma(V, \mathcal{O}_V) \to \widetilde{N}$ for some $(V, \mathcal{O}_V) \in \mathrm{Stone}_{\mathcal{C}}$. The map $(V, \mathcal{O}_V) \to (*, N)$ is a covering, so there exists some point $v \in V$ for which the map of models $N \to \mathcal{O}_{V,v}$ is elementary. Our assumption that s belongs to the equalizer $\mathrm{Eq}(\mathscr{F}(Y_x, \mathcal{O}_Y|_{Y_x}) \rightrightarrows \mathscr{F}(Z_x, \mathcal{O}_Z|_{Z_x}))$ then implies that the stalks $s_y = \psi(\overline{s})_y$ and $s_{y'}$ have the same image in $\mathscr{F}(\{v\}, \mathcal{O}_{V,v})$. It follows that $\psi(\overline{s})_{y'}$ and $s_{y'}$ have the same image in $\mathscr{F}(\{v\}, \mathcal{O}_{V,v})$. Since the composite map $\mathcal{O}_{Y,y'} \to N \to \mathcal{O}_{V,v}$ is elementary, assumption (d) guarantees the injectivity of $\mathscr{F}(\{y\}, \mathcal{O}_{Y,y'}) \to \mathscr{F}(\{v\}, \mathcal{O}_{V,v})$, so that we must have $\psi(\overline{s})_{y'} = s_{y'}$, as desired.