Lecture 18X-Coverings in $\mathrm{Stone}_{\mathcal{C}}$

April 6, 2018

Let $\mathcal{C}$ be an essentially small pretopos, which we regard as fixed throughout this lecture. In Lecture 16X, we saw that the global sections functor induces an equivalence of categories

$$\mathrm{Stone}_{\mathcal{C}} \to \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$$

$$(X, \mathcal{O}_X) \mapsto \Gamma(X; \mathcal{O}_X).$$

The category $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ is equipped with a Grothendieck topology, where a collection of morphisms $\{P_i \to Q\}_{i \in I}$ is a covering if there exists some finite subset $I_0 \subseteq I$ such that the induced map

$$\coprod_{i \in I_0} P_i \to Q$$

is an effective epimorphism in $\mathrm{Pro}(\mathcal{C})$. Transporting this Grothendieck topology along the equivalence $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}) \simeq \mathrm{Stone}_{\mathcal{C}}$, we obtain a Grothendieck topology on the category $\mathrm{Stone}_{\mathcal{C}}$. Our goal in this lecture is to describe this topology more explicitly. Note that a collection of maps

$$\{(X_i, \mathcal{O}_{X_i}), (Y, \mathcal{O}_Y)\}_{i \in I}$$

in $\mathrm{Stone}_{\mathcal{C}}$ is a covering if and only if, for some finite subset $I_0 \subseteq I$, the single map

$$(\amalg_{i \in I_0} X_i, \mathcal{O}_{\amalg_{i \in I_0} X_i}) \to (Y, \mathcal{O}_Y)$$

is a covering (where $\mathcal{O}_{\amalg_{i \in I_0} X_i}$ denotes the sheaf whose restriction to each X_i is given by $\mathcal{O}_{X_i}$). It will therefore be enough to characterize the singleton coverings. First, we need to introduce a bit of (nonstandard) terminology.

Proposition 1. *Let $f : M \to N$ be a morphism of models of $\mathcal{C}$. The following conditions are equivalent:*

(1) *For every object $C \in \mathcal{C}$, the map $M(C) \to N(C)$ is injective.*

(2) *For every object $C \in \mathcal{C}$ and every subobject $C_0 \subseteq C$, the diagram*

$$\begin{array}{ccc} M(C_0) & \longrightarrow & N(C_0) \\ \downarrow & & \downarrow \\ M(C) & \longrightarrow & N(C) \end{array}$$

is a pullback square (in the category of sets).

Proof. Suppose that condition (2) is satisfied. Then, for each object $C \in \mathcal{C}$, the diagram

$$\begin{array}{ccc} M(C) & \longrightarrow & N(C) \\ \downarrow & & \downarrow \\ M(C \times C) & \longrightarrow & N(C \times C) \end{array}$$

is a pullback square, which is equivalent to the injectivity of the map $M(C) \to N(C)$.

We now prove the converse. Let C be an object of $\mathcal{C}$ and let $C_0 \subseteq C$ be a subobject. Let R be the equivalence relation on $(C \amalg \mathbf{1})$ given by the join of C (embedded diagonally in $C \times C$), $C_0 \times C_0$, $C_0 \times \mathbf{1}$, $\mathbf{1} \times C_0$, and $\mathbf{1} \times \mathbf{1}$. Since $\mathcal{C}$ is a pretopos, the equivalence relation R is effective and we can form the quotient $D = (C \amalg \mathbf{1})/R$. The object D can be described more simply as the pushout $C \amalg_{C_0} \mathbf{1}$, and this pushout is preserved by any morphism of pretopoi. In particular, for each model M of $\mathcal{C}$, we can identify $M(D)$ with the pushout $M(C) \amalg_{M(C_0)} \{*\}$ obtained by from $M(C)$ by collapsing the subset $M(C_0) \subseteq M(C)$ to a point.

Suppose that $f : M \to N$ is a map of models, so that f induces a map $f_D : M(D) \to N(D)$. Let us abuse notation by writing $*$ for the distinguished point in both $M(D)$ and $N(D)$. Then we have a canonical bijection

$$f_D^{-1}\{*\} \simeq \{*\} \cup \{x \in M(C) \setminus M(C_0) : f_C(x) \in N(C_0)\}.$$

Consequently, if f_D is injective, then the set $\{x \in M(C) \setminus M(C_0) : f_C(x) \in N(C_0)\}$ is empty, which is equivalent to the statement that the diagram

$$\begin{array}{ccc} M(C_0) & \longrightarrow & N(C_0) \\ \downarrow & & \downarrow \\ M(C) & \longrightarrow & N(C) \end{array}$$

is a pullback. □

Definition 2. Let M and N be models of $\mathcal{C}$. We will say that a morphism $f : M \to N$ is *elementary* if it satisfies the equivalent conditions of Proposition 1.

Remark 3. The terminology of Definition 2 is intended to evoke the notion of an *elementary embedding* in the setting of model theory: we can think of the subobject $C_0 \subseteq C$ as a "proposition" that can hold of elements of $M(C)$, where M is a model of $\mathcal{C}$; elementary morphisms are those which preserve the truth of such propositions.

Example 4. Let $\mathcal{C}$ be a Boolean pretopos. Then *every* morphism $f : M \to N$ in $\mathrm{Mod}(\mathcal{C})$ is elementary. Given $C_0 \subseteq C$ in $\mathcal{C}$, we tautologically have an inclusion

$$M(C_0) \subseteq N(C_0) \times_{N(C)} M(C);$$

the reverse inclusion follows by applying the same argument to a complement of C_0 in C.

Warning 5. Let $\mathcal{C}$ be a pretopos and let $\mathcal{C}'$ be its Booleanization. Then every model M of $\mathcal{C}$ can be promoted (in an essentially unique way) to a model of $\mathcal{C}'$. Let us abuse notation by denoting this model also by M. For every pair of models $M, N \in \mathrm{Mod}(\mathcal{C})$, we have a canonical injection

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C}')}(M, N) \to \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(M, N).$$

If $f : M \to N$ is a morphism in $\mathrm{Mod}(\mathcal{C})$ which belongs to the image of this map, then f is necessarily elementary. However, the converse need not be true (so the terminology of Definition 2 is perhaps misleading).

The relevance of Definition 2 for us is the following:

Theorem 6. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism in $\mathrm{Stone}_{\mathcal{C}}$. The following conditions are equivalent:*

(1) *The induced map $\Gamma(X; \mathcal{O}_X) \to \Gamma(Y; \mathcal{O}_Y)$ is an effective epimorphism in $\mathrm{Pro}(\mathcal{C})$.*

(2) *For each point $y \in Y$, there exists a point $x \in X$ such that $f(x) = y$ and the induced map of models $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is elementary.*

Corollary 7. *Suppose that the pretopos $\mathcal{C}$ is Boolean. Then a map $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ induces an effective epimorphism in* $\mathrm{Pro}(\mathcal{C})$ *if and only if the underlying map of topological spaces $X \to Y$ is surjective.*

Proof. Combine Theorem 6 with Example 4. □

Proof of Theorem 6. We proceed in several steps. Let $f : P \to Q$ be an arbitrary morphism in $\mathrm{Pro}(\mathcal{C})$. Consider the following assertion:

(*i*) The map $f : P \to Q$ is an effective epimorphism in $\mathrm{Pro}(\mathcal{C})$.

We claim that (*i*) is equivalent to the following:

(*ii*) For every monomorphism $U \hookrightarrow V$ in $\mathrm{Pro}(\mathcal{C})$ and every commutative diagram

$$\begin{array}{ccc} P & \longrightarrow & U \\ \Big\downarrow{\scriptstyle f} & \nearrow & \Big\downarrow \\ Q & \longrightarrow & V, \end{array}$$

there exists a dotted arrow as indicated.

The implication (*i*) ⇒ (*ii*) is clear. Conversely, if (*ii*) is satisfied for the inclusion $\mathrm{Im}(u) \hookrightarrow Q$, then assertion (*i*) follows. We saw in Lecture 14X that every monomorphism in $\mathrm{Pro}(\mathcal{C})$ can be realized as a filtered inverse limit of monomorphisms in $\mathcal{C}$. Consequently, (*ii*) is equivalent to the following *a priori* weaker condition:

(*iii*) For every object $C \in \mathcal{C}$ and every subobject $C_0 \subseteq C$, and every commutative diagram

$$\begin{array}{ccc} P & \longrightarrow & C_0 \\ \Big\downarrow & \nearrow & \Big\downarrow \\ Q & \longrightarrow & C, \end{array}$$

there exists a dotted arrow as indicated.

Let us now suppose that $P = \Gamma(X; \mathcal{O}_X)$ and $Q = \Gamma(Y; \mathcal{O}_Y)$ for some objects $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y) \in \mathrm{Stone}_{\mathcal{C}}$. Unwinding the definitions, we can rephrase (*iii*) as follows:

(*iv*) Let $C \in \mathcal{C}$ be an object, let $s_Y \in \Gamma(Y; \mathcal{O}_Y)(C) = \mathcal{O}_Y^C(Y)$, and let $s_X \in \mathcal{O}_X^C(X)$ be the image of s_Y. Suppose that C_0 is a subobject of C and that s_X can be lifted to a global section of the subsheaf $\mathcal{O}_X^{C_0} \subseteq \mathcal{O}_X^C$. Then s_Y can be lifted to a global section of the subsheaf $\mathcal{O}_Y^{C_0} \subseteq \mathcal{O}_Y^C$.

We can restate (*v*) in contrapositive form:

(*v*) Let $C \in \mathcal{C}$ be an object and let $C_0 \subseteq C$ be a subobject. Suppose we are given a global section $s_Y \in \mathcal{O}_Y^C(Y)$ having image $s_X \in \mathcal{O}_X^C(X)$. If there exists a point $y \in Y$ such that the stalk $s_{Y,y}$ does not belong to $\mathcal{O}_{Y,y}^{C_0}$, then there exists a point $x \in X$ such that $s_{X,x}$ does not belong to $\mathcal{O}_{X,x}^{C_0}$.

Note that assertion (*v*) follows immediately from (2) (it suffices to choose $x \in X$ for which $f(x) = y$ and the induced map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is an elementary morphism in $\mathrm{Mod}(\mathcal{C})$). We will complete the proof by showing that assertion (*v*) implies (2).

Assume that (*iv*) is satisfied, and fix a point $y \in Y$. We wish to show that there exists a point $x \in X$ such that $f(x) = y$ and the induced map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is elementary. Suppose otherwise. Then, for each point $x \in f^{-1}(y)$, the induced map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is not elementary. It follows that we can choose an object $C(x) \in \mathcal{C}$, a subobject $C_0(x) \subseteq C(x)$, and an element of $\eta_x \in \mathcal{O}_{Y,y}^{C(x)}$ which does not belong to $\mathcal{O}_{Y,y}^{C_0(x)}$, but the image of η_x in $\mathcal{O}_{X,x}^{C(x)}$ belongs to $\mathcal{O}_{X,x}^{C_0(x)}$. Let U_x be an open neighborhood of x in $f^{-1}\{y\}$ for which the image of η_x in $\mathcal{O}_{X,x'}^{C(x)}$ belongs to $\mathcal{O}_{Y,y}^{C_0(x)}$, for each $x' \in U_x$. Since the fiber $f^{-1}\{y\}$ is compact, we can choose finitely many points $x_1, \ldots, x_n \in f^{-1}\{y\}$ for which the open sets $U_{x_1}, U_{x_2}, \ldots, U_{x_n}$ cover the fiber $f^{-1}\{y\}$. Set $C = C(x_1) \times \cdots \times C(x_n)$, and let $C_0 \subseteq C$ be the union of the subobjects $C_0(x_i) \times \prod_{j \neq i} C(x_j)$. Then we can identify $\{\eta_{x_i}\}_{1 \leq i \leq n}$ with a point $\eta \in \mathcal{O}_{Y,y}^{C}$. By construction, η does not belong to $\mathcal{O}_{Y,y}^{C_0}$, but the image of η in $\mathcal{O}_{X,x}^{C}$ belongs to $\mathcal{O}_{X,x}^{C_0}$ for each $x \in f^{-1}\{y\}$.

Choose a lift of η to a point $s_V \in \mathcal{O}_Y^C(V)$, for some open neighborhood V of Y. Let $s_{f^{-1}(V)}$ denote the image of V in $\mathcal{O}_X^C(f^{-1}(V))$. Then there is a largest open subset $W \subseteq f^{-1}(V)$ for which the restriction $s_{f^{-1}(V)}|_W$ is a section of the subsheaf $\mathcal{O}_X^{C_0} \subseteq \mathcal{O}_X^C$. By construction, the open set W contains $f^{-1}\{y\}$. Since f is a proper map, we can choose a smaller open set $V' \subseteq V$ such that $y \in V'$ and $f^{-1}(V') \subseteq W$. Replacing V by V', we can assume that $s_{f^{-1}(V)}$ belongs to $\mathcal{O}_X^{C_0}(f^{-1}(V))$.

Shrinking V further if necessary, we can arrange that V is both open and closed. In this case, we can extend s_V to a global section s_Y of the sheaf $\mathcal{O}_Y^{C \amalg \mathbf{1}} \simeq \mathcal{O}_Y^C \amalg \underline{\mathbf{1}}$ (which is equal to s_V on the open set V, and carries the complement of V to the second summand of $\mathcal{O}_Y^{C \amalg \mathbf{1}}$). Replacing C by the coproduct $C \amalg \mathbf{1}$ and C_0 by the coproduct $C_0 \amalg \mathbf{1}$, we can assume that $V = Y$: that is, that s_V is a global section of $\mathcal{O}_Y^C$. It then follows from (v) (or (iv)) that s_V is also a global section of the subsheaf $\mathcal{O}_Y^{C_0} \subseteq \mathcal{O}_Y^C$, contradicting our choice of η. □

Corollary 8 (Amalgamation)**.** *Let $f : M \to N$ be an arbitrary morphism in* $\mathrm{Mod}(\mathcal{C})$*, and let $g : M \to M'$ be an elementary morphism in* $\mathrm{Mod}(\mathcal{C})$*. Then there exists a commutative diagram*

$$\begin{array}{ccc} M & \xrightarrow{g} & M' \\ \downarrow{\scriptstyle f} & & \downarrow \\ N & \xrightarrow{g'} & N' \end{array}$$

in $\mathrm{Mod}(\mathcal{C})$*, where g' is also elementary.*

Proof. Let us regard g as a morphism $(*, M') \to (*, M)$ in the category $\mathrm{Stone}_{\mathcal{C}}$. By virtue of Theorem 6, this is a covering (that is, it induces an effective epimorphism in $\mathrm{Pro}(\mathcal{C})$). We can therefore choose a commutative diagram

$$\begin{array}{ccc} (*, M) & \longleftarrow & (*, M') \\ \uparrow & & \uparrow \\ (*, N) & \longleftarrow & (X, \mathcal{O}_X) \end{array}$$

in $\mathrm{Stone}_{\mathcal{C}}$, where the bottom horizontal map is also a covering. Using Theorem 6 again, we can choose a point $x \in X$ for which the induced map $N \to \mathcal{O}_{X,x}$ is elementary. We then obtain a diagram in the category $\mathrm{Mod}(\mathcal{C})$

$$\begin{array}{ccc} M & \xrightarrow{g} & M' \\ \downarrow{\scriptstyle f} & & \downarrow \\ N & \xrightarrow{g'} & \mathcal{O}_{X,x} \end{array}$$

with the desired properties. □