Lecture 17X: Deligne's Theorem; Continuity of Sheaves

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Throughout this lecture, we fix an essentially small pretopos $\mathcal{C}$. In the previous lecture, we showed that the construction

$$(X, \mathcal{O}_X) \mapsto \Gamma(X; \mathcal{O}_X)$$

induces an equivalence of categories $\mathrm{Stone}_{\mathcal{C}} \simeq \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$. Let us note the following immediate consequence:

Theorem 1 (Deligne's Completeness Theorem)**.** *Let $U \subsetneq \mathbf{1}$ be a subobject of the final object of $\mathcal{C}$, which is not equal to $\mathbf{1}$. Then there exists a model M of $\mathcal{C}$ such that $M(U) = \emptyset$.*

Proof. In Lecture 15X, we showed that there exists an effective epimorphism $q : P \to \mathbf{1}$ in $\mathrm{Pro}(\mathcal{C})$, where P is weakly projective. Using Lecture 16X, we can assume that $P = \Gamma(X; \mathcal{O}_X)$, where X is a Stone space and $\mathcal{O}_X$ is an X-model of $\mathcal{C}$. Then $\mathcal{O}_X^U$ can be identified with a subobject of the final object of $\mathrm{Shv}(X)$: that is, with an open subset $V \subseteq X$. Since q is an effective epimorphism, it cannot factor through U. It follows that we must have $V \neq X$. We can therefore choose a point $x \in X$ which does not belong to V. In this case, the stalk $\mathcal{O}_{X,x}^U$ is empty, so that $M = \mathcal{O}_{X,x}$ is a model of $\mathcal{C}$ satisfying $M(U) = \emptyset$. □

Recall that the topos $\mathrm{Shv}(\mathcal{C})$ can be identified with the full subcategory of $\mathrm{Shv}(\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}))$ spanned by those sheaves

$$\mathscr{F} : \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$$

which carry filtered limits in the category $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ to filtered colimits in $\mathcal{S}\mathrm{et}$. To parse this condition, we need to understand filtered limits in the category $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}) \simeq \mathrm{Stone}_{\mathcal{C}}$. These can be described in two different ways:

- Filtered limits in $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ correspond to filtered colimits in the opposite category $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})^{\mathrm{op}}$, which we can identify with the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ spanned by those functors which preserve finite limits and effective epimorphisms. The collection of such functors is closed under filtered colimits, so that filtered colimits in $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})^{\mathrm{op}}$ are computed "pointwise."

- If we identify $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ with the category $\mathrm{Stone}_{\mathcal{C}}$, then filtered limits can be understood "geometrically." Suppose we are given a diagram $\{(X_\alpha, \mathcal{O}_{X_\alpha})\}$ in the category $\mathrm{Stone}_{\mathcal{C}}$, indexed by (the opposite of) a filtered category. The limit of this diagram is given by $(X, \mathcal{O}_X)$, where $X = \varprojlim_\alpha X_\alpha$ is the limit of the diagram $\{X_\alpha\}$ in the category of topological spaces, and $\mathcal{O}_X^C \simeq \varinjlim_\alpha \pi_\alpha^* \mathcal{O}_{X_\alpha}^C$ for each object $C \in \mathcal{C}$ (here $\pi_\alpha : X \to X_\alpha$ denotes the projection map). To check this, the main ingredient we need is the observation that $\mathcal{O}_X$ is again an X-model of $\mathcal{C}$: that is, for every point $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a model of $\mathcal{C}$. This is clear: as a functor from $\mathcal{C}$ to the category of sets, the stalk $\mathcal{O}_{X,x}$ can be identified with the filtered colimit of the diagram $\{\mathcal{O}_{X_\alpha, x_\alpha}\}$ (where $x_\alpha = \pi_\alpha(x) \in X_\alpha$); this is a model of $\mathcal{C}$, since the subcategory $\mathrm{Mod}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ is closed under filtered colimits.

Remark 2. From the equivalence of these two descriptions of filtered limits, we can extract the following consequence: for any filtered diagram $\{(X_\alpha, \mathcal{O}_{X_\alpha})\}$ in the category $\mathrm{Stone}_{\mathcal{C}}$ having limit $(X, \mathcal{O}_X)$ and any object $C \in \mathcal{C}$, the canonical map

$$\varinjlim \mathcal{O}_{X_\alpha}^C(X_\alpha) \to \mathcal{O}_X^C(X)$$

is bijective.

Example 3. Let $\mathcal{C}$ be the category of coherent objects of the classifying topos $\mathrm{Fun}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}, \mathcal{S}\mathrm{et})$, and let $E \in \mathcal{C}$ be the inclusion functor $\mathcal{S}\mathrm{et}_{\mathrm{fin}} \hookrightarrow \mathcal{S}\mathrm{et}$. Then the construction $(X, \mathcal{O}_X) \mapsto (X, \mathcal{O}\, X^E)$ induces an equivalence from $\mathrm{Stone}_{\mathcal{C}}$ with the category of pairs $(X, \mathscr{G})$, where X is a Stone space and $\mathscr{G}$ is a sheaf of sets on X. In this case, Remark 2 asserts that for any filtered diagram $\{(X_\alpha, \mathscr{G}_\alpha)\}$ in this category, the canonical map

$$\varinjlim_{\alpha} \mathscr{G}_\alpha(X_\alpha) \to (\varinjlim_{\alpha} \pi_\alpha^* \mathscr{G}_\alpha)(X)$$

is bijective, where $X = \varprojlim X_\alpha$ and $\pi_\alpha : X \to X_\alpha$ denote the projection maps. From this, Remark 2 is immediate.

Exercise 4. Show that the map

$$\varinjlim_{\alpha} \mathscr{G}_\alpha(X_\alpha) \to (\varinjlim_{\alpha} \pi_\alpha^* \mathscr{G}_\alpha)(X)$$

is bijective more generally under the assumption that each X_α is a compact Hausdorff space (in which case X has the same properties).

Example 5. Let $\mathrm{Mod}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ be the category of models of $\mathcal{C}$. Then the opposite category $\mathrm{Mod}(\mathcal{C})^{\mathrm{op}}$ can be identified with the full subcategory of $\mathrm{Stone}_{\mathcal{C}}$ spanned by those pairs $(X, \mathcal{O}_X)$ where the Stone space X is a single point. As a subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$, the category $\mathrm{Mod}(\mathcal{C})$ is closed under filtered colimits. In particular, the category $\mathrm{Mod}(\mathcal{C})$ admits filtered colimits, so the opposite category $\mathrm{Mod}(\mathcal{C})^{\mathrm{op}}$ admits filtered limits. These are preserved by the inclusion $\mathrm{Mod}(\mathcal{C})^{\mathrm{op}} \hookrightarrow \mathrm{Stone}_{\mathcal{C}}$.

Example 6. Let X be a Stone space and let $\mathcal{O}_X$ be an X-model of $\mathcal{C}$. For every point $x \in X$, we have a filtered diagram $\{(U, \mathcal{O}_X|_U)\}_{x \in U}$ in the category $\mathrm{Stone}_{\mathcal{C}}$, where U ranges over all *clopen* neighborhoods of x. This diagram admits an inverse limit, given by $(\{x\}, \mathcal{O}_{X,x})$, which belong to the essential image of the inclusion $\mathrm{Mod}(\mathcal{C})^{\mathrm{op}} \hookrightarrow \mathrm{Stone}_{\mathcal{C}}$.

It turns out that the filtered diagrams appearing in Examples 5 and 6 are the only ones that we need to consider.

Theorem 7. *Let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a functor which carries finite coproducts in $\mathrm{Stone}_{\mathcal{C}}$ to products in $\mathcal{S}\mathrm{et}$ (this condition is automatically satisfied if $\mathscr{F}$ corresponds to a sheaf on the category $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$). Then $\mathscr{F}$ carries filtered limits in $\mathrm{Stone}_{\mathcal{C}}$ to filtered colimits in $\mathcal{S}\mathrm{et}$ if and only if it satisfies the following pair of conditions:*

- (a) *The composite functor $\mathrm{Mod}(\mathcal{C}) \hookrightarrow \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \xrightarrow{\mathscr{F}} \mathcal{S}\mathrm{et}$ preserves filtered colimits.*
- (b) *For every object $(X, \mathcal{O}_X)$ in $\mathrm{Stone}_{\mathcal{C}}$ and every point $x \in X$, the canonical map*

$$\varinjlim_{x \in U} \mathscr{F}(U, \mathcal{O}_X|_U) \to \mathscr{F}(\{x\}, \mathcal{O}_{X,x})$$

is a bijection. Here the colimit is taken over all clopen neighborhoods of x in X.

Proof. The necessity of conditions (a) and (b) follows from Examples 5 and 6. Let us therefore assume that (a) and (b) are satisfied, and consider an arbitrary filtered diagram $\{(X_\alpha, \mathcal{O}_{X_\alpha})\}$ in the category $\mathrm{Stone}_{\mathcal{C}}$. Set $X = \varprojlim X_\alpha$ and, for each index α, let $\pi_\alpha : X \to X_\alpha$ denote the projection map and let $\mathcal{O}_X$ be the X-model of $\mathcal{C}$ given by the formula $\mathcal{O}_X^C = \varinjlim \pi_\alpha^* \mathcal{O}_{X_\alpha}^C$ for $C \in \mathcal{C}$.

Let $\mathcal{U}_0(X)$ denote the collection of clopen subsets of X and define $\mathcal{U}_0(X_\alpha)$ similarly. We define functors

$$\mathscr{G} : \mathcal{U}_0(X)^{\mathrm{op}} \to \mathcal{S}\mathrm{et} \qquad \mathscr{G}_\alpha : \mathcal{U}_0(X_\alpha)^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$$

by the formulae

$$\mathscr{G}(U) = \mathscr{F}(U, \mathcal{O}_X|_U) \qquad \mathscr{G}_\alpha(U) = \mathscr{F}(U, \mathcal{O}_{X_\alpha}|_U).$$

Using our assumption that $\mathscr{F}$ carries finite coproducts in $\mathrm{Stone}_{\mathcal{C}}$ to finite products in the category of sets, we see that the presheaf $\mathscr{G}$ carries finite disjoint unions in $\mathcal{U}_0(X)$ to products of sets. It follows that $\mathscr{G}$ admits an essentially unique extension to a sheaf of sets on X, which we will also denote by $\mathscr{G}$. Similarly, each $\mathscr{G}_\alpha$ can be extended to a sheaf of sets on X_α. Moreover, the map $(X, \mathcal{O}_X) \to (X_\alpha, \mathcal{O}_{X_\alpha})$ in $\mathrm{Stone}_{\mathcal{C}}$ induces a map of sheaves $f_\alpha : \pi_\alpha^* \mathscr{G}_\alpha \to \mathscr{G}$. These maps are compatible as α varies, and induce a map

$$f_\infty : \varinjlim_\alpha \pi_\alpha^* \mathscr{G}_\alpha \to \mathscr{G}$$

in the category $\mathrm{Shv}(X)$ of $\mathcal{S}\mathrm{et}$-valued sheaves on X.

We wish to show that the canonical map

$$\varinjlim_\alpha \mathscr{F}(X_\alpha, \mathcal{O}_{X_\alpha}) \to \mathscr{F}(X, \mathcal{O}_X)$$

is a bijection. Unwinding the definitions, we can write this map as a composition

$$\varinjlim_\alpha \mathscr{G}_\alpha(X_\alpha) \to (\varinjlim_\alpha \pi_\alpha^* \mathscr{G}_\alpha)(X) \xrightarrow{f_\infty} \mathscr{G}(X).$$

Here the first map is bijective by virtue of Example 3. To complete the proof, it will suffice to show that f_∞ is an isomorphism of sheaves on X. For this, it will suffice to show that for each point $x \in X$, the induced map of stalks

$$\varinjlim_\alpha (\pi_\alpha^* \mathscr{G}_\alpha)_x = \varinjlim_\alpha \mathscr{G}_{\alpha, x_\alpha} \to \mathscr{G}_x$$

is a bijection; here x_α denotes the point $\pi_\alpha(x) \in X_\alpha$. Using assumption (b), we can rewrite this map as

$$\varinjlim_\alpha \mathscr{F}(\{x_\alpha\}, \mathcal{O}_{X_\alpha, x_\alpha}) \to \mathscr{F}(\{x\}, \mathcal{O}_{X,x}).$$

This map is bijective by virtue of (a) (and the construction of $\mathcal{O}_X$). □

Combining Theorem 7 with the results of Lecture 15X, we obtain the following:

Corollary 8. *There is a fully faithful embedding* $\mathrm{Shv}(\mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$*, whose essential image consists of those functors* $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ *which are sheaves (with respect to the topology given by finite coverings in* $\mathrm{Pro}(\mathcal{C})$*) and which satisfy conditions* (a) *and* (b) *of Theorem 7.*