Lecture 16X: X-Models

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As we have seen, topos theory provides a simultaneous generalization of point-set topology and first-order logic. We now take advantage of this.

Definition 1. Let $\mathcal{C}$ be an essentially small pretopos, which we regard as fixed throughout this lecture. Let X be a topological space. An *X-model of* $\mathcal{C}$ is a geometric morphism of topoi from $\mathrm{Shv}(X)$ to $\mathrm{Shv}(\mathcal{C})$.

Let us unwind Definition 1. As we saw in Lecture 13, giving a geometric morphism from $\mathrm{Shv}(X)$ to $\mathrm{Shv}(\mathcal{C})$ is equivalent to giving a functor $\mathcal{O}_X : \mathcal{C} \to \mathrm{Shv}(X)$ which preserves finite limits, effective epimorphisms, and finite coproducts. We will denote the value of $\mathcal{O}_X$ on an object $C \in \mathcal{C}$ by $\mathcal{O}_X^C$. Here $\mathcal{O}_X^C$ denotes a sheaf of sets on X which we can evaluate on an open subset $U \subseteq X$ to obtain a set $\mathcal{O}_X^C(U)$. We can also take the *stalk* of $\mathcal{O}_X^C$ at a point $x \in X$, to obtain a set $\mathcal{O}_{X,x}^C = \varinjlim_{x \in U} \mathcal{O}_X^C(U)$. We can therefore reformulate Definition 1 as follows:

Definition 2. Let X be a topological space. An *X-model of* $\mathcal{C}$ is a functor

$$\mathcal{O}_X : \mathcal{C} \times \mathcal{U}(X)^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$$

with the following properties:

- (a) For each $C \in \mathcal{C}$, the functor $U \mapsto \mathcal{O}_X^C(U)$ is a sheaf of sets on X. Given an element $s \in \mathcal{O}_X^C(U)$ and an open subset $V \subseteq U$, we let $s|_V$ denote the image of s in $\mathcal{O}_X^C(V)$.
- (b) For each open set $U \subseteq X$, the functor $C \mapsto \mathcal{O}_X^C(U)$ preserves finite limits.
- (c) Given an effective epimorphism $C' \to C$ in the pretopos $\mathcal{C}$, the induced map $\mathcal{O}_X^{C'} \to \mathcal{O}_X^C$ is an effective epimorphism of sheaves. That is, given a section $s \in \mathcal{O}_X^C(U)$ for some open set $U \subseteq X$, we can choose an open covering $\{U_\alpha\}$ of U such that each $s|_{U_\alpha}$ can be lifted to an element of $\mathcal{O}_X^{C'}(U_\alpha)$.
- (d) For every finite collection of objects $\{C_i\}_{i \in I}$ in $\mathcal{C}$ having coproduct C, we can regard $\mathcal{O}_X^C$ as the coproduct of $\mathcal{O}_X^{C_i}$ in the category $\mathrm{Shv}(X)$. In other words, giving an element $s \in \mathcal{O}_X^C(U)$ is equivalent to giving a decomposition $U = \amalg_{i \in I} U_i$ and a collection of elements $s_i \in \mathcal{O}_X^{C_i}(U_i)$.

Remark 3. In the situation of Definition 2, conditions (b), (c) and (d) can also be formulated stalkwise as follows:

- (b') For every point $x \in X$, the functor $C \mapsto \mathcal{O}_{X,x}^C$ preserves finite limits.
- (c') For every point $x \in X$, the functor $C \mapsto \mathcal{O}_{X,x}^C$ carries effective epimorphisms in $\mathcal{C}$ to surjections of sets.
- (d') For every point $x \in X$, the functor $C \mapsto \mathcal{O}_{X,x}^C$ preserves finite coproducts.

Together, these conditions assert that for each point $x \in X$, the functor $C \mapsto \mathcal{O}_{X,x}^C$ is a model of $\mathcal{C}$. We will denote this model by $\mathcal{O}_{X,x}$.

Roughly speaking, we can think of an X-model $\mathcal{O}_X$ of $\mathcal{C}$ as a family of models $\{\mathcal{O}_{X,x}\}_{x \in X}$ of $\mathcal{C}$, which in some sense depend *continuously* on the point $x \in X$.

The following observation will be useful for verifying condition (d) of Definition 2:

Lemma 4. *Let $\mathcal{C}$ and $\mathcal{D}$ be pretopoi and let $f : \mathcal{C} \to \mathcal{D}$ be a left exact functor. The following conditions are equivalent:*

(1) *The functor f preserves finite coproducts.*

(2) *The functor f preserves initial objects, and the canonical map $f(\mathbf{1}) \amalg f(\mathbf{1}) \to f(\mathbf{1} \amalg \mathbf{1})$ is an isomorphism.*

Proof. The implication (1) $\Rightarrow$ (2) is immediate. Conversely, suppose that (2) is satisfied. To show that f preserves finite coproducts. Since f preserves empty coproducts, it will suffice to show that for every pair of objects $C, D \in \mathcal{C}$, the canonical map $f(C) \amalg f(D) \to f(C \amalg D)$ is an equivalence. Note that we have a canonical map $f(C \amalg D) \to f(\mathbf{1} \amalg \mathbf{1})$. Using condition (2), we are reduced to showing that the maps

$$(f(C) \amalg f(D)) \times_{f(\mathbf{1} \amalg \mathbf{1})} f(\mathbf{1}) \to (f(C \amalg D)) \times_{f(\mathbf{1} \amalg \mathbf{1})} f(\mathbf{1})$$

are isomorphisms (for each of the summand inclusions $\mathbf{1} \to \mathbf{1} \amalg \mathbf{1}$.). By symmetry, it suffices to treat the case of the inclusion of the first summand. Using the assumption that coproducts are pullback-stable in $\mathcal{D}$ and that f preserves finite limits, we can rewrite this map as

$$f(C \times_{\mathbf{1} \amalg \mathbf{1}} \mathbf{1}) \amalg f(D \times_{\mathbf{1} \amalg \mathbf{1}} \mathbf{1}) \to f((C \amalg D) \times_{\mathbf{1} \amalg \mathbf{1}} \mathbf{1}).$$

Using the disjointness of coproducts in $\mathcal{C}$, we can further rewrite this as

$$f(C) \amalg f(\emptyset) \to f(C),$$

which is an isomorphism since f preserves initial objects. □

Remark 5 (Functoriality)**.** Let $f : X \to Y$ be a continuous map of topological spaces and let $\mathcal{O}_Y$ be a Y-model of $\mathcal{C}$. Then we can construct an X-model $f^* \mathcal{O}_Y$ of $\mathcal{C}$, given concretely by the formula $(f^* \mathcal{O}_Y)^C = f^* \mathcal{O}_Y^C$.

Note that, if we think of $\mathcal{O}_Y$ as encoding the data of a geometric morphism of topoi $\mathrm{Shv}(Y) \to \mathrm{Shv}(\mathcal{C})$, then $f^* \mathcal{O}_Y$ simply encodes the composite geometric morphism $\mathrm{Shv}(X) \xrightarrow{f} \mathrm{Shv}(Y) \to \mathrm{Shv}(\mathcal{C})$. Note also that the stalk of $(f^* \mathcal{O}_Y)$ at a point $x \in X$ is given by $\mathcal{O}_{Y,f(x)}$

Definition 6. We define a category $\mathrm{Top}_{\mathcal{C}}$ as follows:

- An object of $\mathrm{Top}_{\mathcal{C}}$ consists of a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is an X-model of $\mathcal{C}$.
- A morphism from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ consists of a continuous map of topological spaces $f : X \to Y$ together with a natural transformation of functors $f^* \mathcal{O}_Y \to \mathcal{O}_X$.

Example 7. Let $\mathcal{C} = \mathcal{S}\mathrm{et}_{\mathrm{fin}}$ be the category of finite sets. Then, for every topological space X, there is an essentially unique X-model of $\mathcal{C}$, given by the formula $\mathcal{O}_X^S = \underline{S}$ (here $\underline{S}$ denotes the constant sheaf associated to the finite set S). The construction $(X, \mathcal{O}_X) \mapsto X$ induces an equivalence from the category $\mathrm{Top}_{\mathcal{C}}$ of Definition 6 to the category Top of topological spaces.

Example 8. Let $\mathcal{C}$ be the category of coherent objects of the classifying topos of commutative rings, given by $\mathrm{Fun}(\{\text{Finitely presented commutative rings}\}, \mathcal{S}\mathrm{et})$. Then the datum of an X-model of $\mathcal{C}$ is equivalent to the datum of a sheaf of commutative rings on $\mathcal{C}$, and $\mathrm{Top}_{\mathcal{C}}$ is equivalent to the category of ringed spaces.

Proposition 9. *Let X be a topological space and let $\mathcal{O}_X$ be an X-model of $\mathcal{C}$. We let $\Gamma(X;\mathcal{O}_X) : \mathcal{C} \to \mathcal{S}\mathrm{et}$ denote the functor given by the construction*

$$(C \in \mathcal{C}) \mapsto (\mathcal{O}_X^C(X) \in \mathcal{S}\mathrm{et}).$$

Then:

(1) *The functor $\Gamma(X;\mathcal{O}_X)$ preserves finite limits, and can therefore be regarded as a pro-object of $\mathcal{C}$.*

(2) *If X is a Stone space, then the functor $\Gamma(X;\mathcal{O}_X)$ also preserves effective epimorphisms, and is therefore* weakly projective *as a pro-object of $\mathcal{C}$.*

Proof. Assertion (1) is immediate from part (b) of Definition 2. To prove (2), we note that if $C' \to C$ is an effective epimorphism in $\mathcal{C}$, then part (c) of Definition 2 guarantees that the induced map $\mathcal{O}_X^{C'} \to \mathcal{O}_X^C$ is an effective epimorphism of *sheaves*. In particular, for every global section $s \in \mathcal{O}_X^C(X)$, we can choose an open covering $\{U_\alpha\}$ of X such that each $s|_{U_\alpha}$ lifts to an element $\widetilde{s}_\alpha \in \mathcal{O}_X^{C'}(U_\alpha)$. If X is a Stone space, then the open covering $\{U_\alpha\}$ can be refined to an open covering by *disjoint* open subsets of X, in which case we can amalgamate the sections $\widetilde{s}_\alpha$ to a single element $\widetilde{s} \in \mathcal{O}_X^{C'}(X)$ lying over s. $\square$

Note that if $f : (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ is a morphism in the category $\mathrm{Top}_{\mathcal{C}}$, then we have canonical maps

$$\Gamma(Y;\mathcal{O}_Y)(C) = \mathcal{O}_Y^C(Y) \to (f^*\,\mathcal{O}_Y^C)(X) \to \mathcal{O}_X^C(X) = \Gamma(X;\mathcal{O}_X)(C),$$

depending functorially on C. This determines a natural transformation of functors from $\Gamma(Y;\mathcal{O}_Y)$ to $\Gamma(X;\mathcal{O}_X)$, or equivalently a morphism

$$\Gamma(X;\mathcal{O}_X) \to \Gamma(Y;\mathcal{O}_Y)$$

in the category $\mathrm{Pro}(\mathcal{C})$. In other words, we can regard the construction $(X,\mathcal{O}_X) \mapsto \Gamma(X;\mathcal{O}_X)$ as a functor

$$\Gamma : \mathrm{Top}_{\mathcal{C}} \to \mathrm{Pro}(\mathcal{C}).$$

Notation 10. We let $\mathrm{Stone}_{\mathcal{C}}$ denote the full subcategory of $\mathrm{Top}_{\mathcal{C}}$ spanned by those pairs $(X,\mathcal{O}_X)$ where X is a Stone space. It follows from Proposition 9 that the global sections functor restricts to a functor

$$\Gamma : \mathrm{Stone}_{\mathcal{C}} \to \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}).$$

Theorem 11. *The functor $\Gamma : \mathrm{Stone}_{\mathcal{C}} \to \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ is an equivalence of categories.*

Sketch. We sketch an explicit construction of an inverse to the functor Γ. First, recall that there is an essentially unique pretopos morphism $\iota : \mathcal{S}\mathrm{et}_{\mathrm{fin}} \to \mathcal{C}$, given by $\iota(S) = \coprod_{s\in S} \mathbf{1}$. Precomposition with ι determines a forgetful functor

$$\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}) \to \mathrm{Pro}^{\mathrm{wp}}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}) = \mathrm{Pro}(\mathcal{S}\mathrm{et}_{\mathrm{fin}}) = \mathrm{Stone},$$

where Stone is the category of Stone spaces. Explicitly, if M is an object of $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$, viewed as a functor $M : \mathcal{C} \to \mathcal{S}\mathrm{et}$ which preserves finite limits and effective epimorphisms, then this forgetful functor carries M to a Stone space X characterized by the existence of natural bijections $\mathrm{Hom}_{\mathrm{Top}}(X,S) \simeq M(\iota(S))$. For every integer $n \geq 0$, let $\mathbf{n}$ denote the coproduct $\mathbf{1} \amalg \cdots \amalg \mathbf{1}$ of n copies of the final object of $\mathcal{C}$. We can then restate our characterization of X as follows: for each n, we can identify $M(\mathbf{n})$ with the set of all n-tuples $(U_1,\ldots,U_n)$ of disjoint clopen subsets $U_1,\ldots,U_n \subseteq X$ satisfying $X = U_1 \amalg \cdots \amalg U_n$.

Let M and X be as above, and let $\mathcal{U}_0(X)$ denote the Boolean algebra of clopen subsets of X. For each object $C \in \mathcal{C}$ and each $U \in \mathcal{U}_0(X)$,

$$\mathcal{O}_X^C(U) = M(\mathbf{1} \amalg C) \times_{M(\mathbf{2})} \{(X-U,U)\}.$$

Note that if $V \subseteq U$ is a smaller clopen subset, then we have a pullback square

$$\begin{array}{ccc} \mathbf{1} \amalg C \amalg C & \longrightarrow & \mathbf{1} \amalg C \\ \downarrow & & \downarrow \\ \mathbf{3} & \longrightarrow & \mathbf{2} \end{array}$$

in $\mathcal{C}$, so that $\mathcal{O}_X^C(U)$ can also be identified with the set $M(\mathbf{1} \amalg C \amalg C) \times_{M(\mathbf{3})} \{(X-U, U-V, V)\}$. We therefore have a canonical map

$$\mathcal{O}_X^C(U) = M(\mathbf{1} \amalg C \amalg C) \times_{M(\mathbf{3})} \{(X-U, U-V, V)\} \to M(\mathbf{1} \amalg \mathbf{1} \amalg C) \times_{M(\mathbf{3})} \{(X-U, U-V, V)\} \simeq \mathcal{O}_X^C(V).$$

For fixed $C \in \mathcal{C}$, these maps endow $\mathcal{O}_X^C$ with the structure of a *presheaf of sets* on the poset $\mathcal{U}_0(X)$ (Exercise: check this.)

In fact, we claim that this presheaf is actually a sheaf. Since every open covering of a Stone space can be refined to a finite covering by disjoint clopen sets, it will suffice to prove the following:

- For $V \subseteq U$ as above, the restriction maps $\mathcal{O}_X^C(U) \to \mathcal{O}_X^C(V)$ and $\mathcal{O}_X^C(U) \to \mathcal{O}_X^C(U-V)$ induce a bijection $\mathcal{O}_X^C(U) \to \mathcal{O}_X^C(V) \times \mathcal{O}_X^C(U-V)$. This follows from the fact that the diagram of sets

$$\begin{array}{ccc} M(\mathbf{1} \amalg C \amalg C) & \longrightarrow & M(\mathbf{1} \amalg \mathbf{1} \amalg C) \\ \downarrow & & \downarrow \\ M(\mathbf{1} \amalg C \amalg \mathbf{1}) & \longrightarrow & M(\mathbf{1} \amalg \mathbf{1} \amalg \mathbf{1}) \end{array}$$

  is a pullback square (since M is left exact).

- When $U = \emptyset$, the set $\mathcal{O}_X^C(U)$ is a singleton. We leave this as an exercise.

It follows that that for each object $C \in \mathcal{C}$, the construction $(U \in \mathcal{U}_0(X)) \mapsto \mathcal{O}_X^C(U)$ admits an essentially unique extension to a sheaf of sets on X, which we will also denote by $\mathcal{O}_X^C$. We claim that the functor

$$\mathcal{O}_X : \mathcal{C} \to \mathrm{Shv}(X) \qquad C \mapsto \mathcal{O}_X^C$$

is an X-model of $\mathcal{C}$, in the sense of Definition 2. The verification of (a) was sketched above, and condition (b) follows from our assumption that M preserves finite limits. Since M preserves effective epimorphisms, an effective epimorphism $C' \to C$ induces a surjection $\mathcal{O}_X^{C'}(U) \to \mathcal{O}_X^C(U)$ for every *clopen* subset $U \subseteq X$, and therefore an effective epimorphism of sheaves $\mathcal{O}_X^{C'} \to \mathcal{O}_X^C$; this proves (c). To verify condition (d), we note that for every finite set S and each clopen subset $U \subseteq X$, we have

$$\begin{aligned} \mathcal{O}_X^{\iota S}(U) &= M(\mathbf{1} \amalg \iota(S)) \times_{M(\mathbf{1}\amalg\mathbf{1})} \{(X-U, U)\} \\ &= \{\text{Clopen decompositions } X = \amalg_{s \in S \cup \{0\}} X_s \text{ with } X_0 = X - U\ \} \\ &= \{\text{Clopen decompositions } U = \amalg_{s \in S} U_s\ \}. \end{aligned}$$

so that $\mathcal{O}_X^{\iota S}$ can be identified with the constant sheaf with value S. Condition (d) now follows from Lemma 4.

Summarizing the above discussion, from a functor $M : \mathcal{C} \to \mathcal{S}\mathrm{et}$ which preserves finite limits and effective epimorphisms, we can construct an object $(X, \mathcal{O}_X)$ in $\mathrm{Stone}_{\mathcal{C}}$. Note that for $C \in \mathcal{C}$, we have canonical isomorphisms

$$\begin{aligned} \Gamma(X; \mathcal{O}_X)(C) &= \mathcal{O}_X^C(X) \\ &= M(\mathbf{1} \amalg C) \times_{M(\mathbf{1}\amalg\mathbf{1})} \{(\emptyset, X)\} \\ &\simeq M(C). \end{aligned}$$

since the diagram

$$\begin{array}{ccc} C & \longrightarrow & \mathbf{1} \amalg C \\ \downarrow & & \downarrow \\ \mathbf{1} & \longrightarrow & \mathbf{1} \amalg \mathbf{1} \end{array}$$

is a pullback and $M(\mathbf{1})$ is a singleton. In other words, the construction $M \mapsto (X, \mathcal{O}_X)$ is right inverse to the functor $\Gamma : \mathrm{Stone}_{\mathcal{C}} \to \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ (up to isomorphism).

Consider now the composition in the other direction. Let Y be a Stone space and $\mathcal{O}_Y$ a Y-model of $\mathcal{C}$, and suppose that we apply the above construction to the functor $M = \Gamma(Y, \mathcal{O}_Y)$. For every finite set S, we have $M(\iota(S)) = \mathcal{O}_Y^{\iota(S)}(Y) = \underline{S}(Y)$, where $\underline{S}$ is the constant sheaf on Y with the value S. The value of this sheaf on Y can be identified with the set of continuous maps $Y \to S$, functorially in S. It follows that the Stone space X constructed above is canonically homeomorphic to Y. Let us identify X with Y. If C is an object of $\mathcal{C}$ and U is a clopen subset of X, we have canonical bijections

$$\begin{array}{rcl} \mathcal{O}_X^C(U) & = & M(\mathbf{1} \amalg C) \times_{M(\mathbf{1} \amalg \mathbf{1})} \{(X - U, U)\} \\ & \simeq & \mathcal{O}_Y^{\mathbf{1} \amalg C}(Y) \times_{\mathcal{O}_Y^{\mathbf{1} \amalg \mathbf{1}}(Y)} \{(X - U, U)\} \\ & \simeq & (\underline{1} \amalg \mathcal{O}_Y^C)(Y) \times_{(\underline{1} \amalg \underline{1})(Y)} \{(X - U, U)\}. \end{array}$$

where $\underline{1}$ denotes the final object of $\mathrm{Shv}(Y)$ and coproducts are formed in the category of sheaves; we conclude by observing that this fiber product is canonically isomorphic to the set $\mathcal{O}_Y^C(U)$. $\square$