Lecture 15X: Pro-Étale Sheaves

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Throughout this lecture, we let $\mathcal{C}$ denote an essentially small coherent category with disjoint coproducts (for example, a small pretopos). In the previous lecture, we proved that $\mathrm{Pro}(\mathcal{C})$ is also a coherent category with disjioont coproducts. In particular, we can endow $\mathrm{Pro}(\mathcal{C})$ with a finitary Grothendieck topology, where a finite collection of morphisms $\{U_i \to X\}$ is a covering if the induced map $\coprod U_i \to X$ is an effective epimorphism. We let $\mathrm{Shv}(\mathrm{Pro}(\mathcal{C}))$ denote the category of sheaves with respect to this topology.

Warning 1. The category $\mathrm{Shv}(\mathrm{Pro}(\mathcal{C}))$ is *not* a topos (note that $\mathrm{Pro}(\mathcal{C})$ is not small).

Example 2. Let X be a quasi-compact and quasi-separated scheme, and let $\mathrm{Sch}_X^{\mathrm{et}}$ denote the category of quasi-compact, quasi-separated schemes U equipped with an étale map $U \to X$. Then $\mathrm{Sch}_X^{\mathrm{et}}$ is an essentially small coherent category, and $\mathrm{Shv}(\mathrm{Pro}(\mathrm{Sch}_X^{\mathrm{et}}))$ can be identified with the category of *pro-étale sheaves on X* introduced by Bhatt-Scholze.

Similarly, Scholze's category of pro-étale sheaves on a (quasi-compact, quasi-separated) perfectoid space X can be realized as $\mathrm{Shv}(\mathrm{Pro}(\mathcal{C}))$, where $\mathcal{C}$ is the category of (quasi-compact, quasi-separated) perfectoid spaces which are étale over X.

Our first goal is to understand the relationship of $\mathrm{Shv}(\mathrm{Pro}(\mathcal{C}))$ with the topos $\mathrm{Shv}(\mathcal{C})$.

Proposition 3. *Let $\mathcal{C}$ be as above and let $\mathscr{F} : \mathrm{Pro}(\mathcal{C})^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a functor. Then:*

(1) *If $\mathscr{F}$ is a sheaf on the category $\mathrm{Pro}(\mathcal{C})$, then the restriction $\mathscr{F}|_{\mathcal{C}^{\mathrm{op}}}$ is a sheaf on $\mathcal{C}$.*

(2) *If $\mathscr{F}|_{\mathcal{C}^{\mathrm{op}}}$ is a sheaf on $\mathcal{C}$ and the functor $\mathscr{F}$ commutes with filtered colimits, then $\mathscr{F}$ is a sheaf on $\mathrm{Pro}(\mathcal{C})$.*

Proof. We will prove (2) and leave (1) as an exercise for the reader. Assume that $\mathscr{F}|_{\mathcal{C}^{\mathrm{op}}}$ is a sheaf and that $\mathscr{F}$ commutes with filtered colimits; we wish to show that $\mathscr{F}$ is a sheaf. For this, we must prove the following:

(a) The functor $\mathscr{F}$ carries finite coproducts in $\mathrm{Pro}(\mathcal{C})$ to products of sets.

(b) For each effective epimorphism $U \to X$ in $\mathrm{Pro}(\mathcal{C})$, the diagram

$$\mathscr{F}(X) \to \mathscr{F}(U) \rightrightarrows \mathscr{F}(U \times_X U)$$

is an equalizer.

We begin with (a). Suppose we are given a finite collection of objects $C_1, \ldots, C_n \in \mathrm{Pro}(\mathcal{C})$, each of which is the limit of a pro-system $\{C_{i,\alpha}\}$ in $\mathcal{C}$; without loss of generality, we may assume that each of these pro-systems is indexed by the same category. Then the coproduct $C_1 \amalg \cdots \amalg C_n$ is given by the limit of the pro-system $\{C_{1,\alpha} \amalg \cdots \amalg C_{n,\alpha}\}$. Since $\mathscr{F}$ carries filtered limits in $\mathrm{Pro}(\mathcal{C})$ to filtered colimits of sets, we are reduced to showing that the canonical map

$$\varinjlim_{\alpha} \mathscr{F}(C_{1,\alpha} \amalg \cdots \amalg C_{n,\alpha}) \to \prod_{1 \le i \le n} \varinjlim_{\alpha} \mathscr{F}(C_{i,\alpha})$$

is an isomorphism, which follows from the fact that filtered colimits of sets commute with products and our assumption that $\mathscr{F}|_{\mathcal{C}^{\mathrm{op}}}$ is a sheaf.

We now prove (b). Let $f : U \to X$ be an effective epimorphism in $\mathrm{Pro}(\mathcal{C})$. Then we can write f as the limit of a diagram $\{f_\alpha : U_\alpha \to X_\alpha$ of effective epimorphisms in $\mathcal{C}$. Using our assumption that $\mathscr{F}$ is compatible with filtered limits in $\mathrm{Pro}(\mathcal{C})$, we are reduced to showing that the diagram

$$\varinjlim_\alpha \mathscr{F}(X_\alpha) \to \varinjlim_\alpha \mathscr{F}(U_\alpha) \rightrightarrows \varinjlim_\alpha \mathscr{F}(U_\alpha \times_{X_\alpha} U_\alpha).$$

This follows from our assumption that $\mathscr{F}|_{\mathcal{C}^{\mathrm{op}}}$ is a sheaf, since the collection of equalizer diagrams in $\mathcal{S}\mathrm{et}$ is closed under filtered colimits. □

The universal property of $\mathrm{Pro}(\mathcal{C})$ implies that any presheaf $\mathscr{F}_0 \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ admits an essentially unique extension to a presheaf $\mathscr{F} \in \mathrm{Fun}(\mathrm{Pro}(\mathcal{C})^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ which preserves filtered colimits. It follows from Proposition 3 that $\mathscr{F}_0$ is a sheaf if and only if $\mathscr{F}$ is a sheaf. This proves the following:

Proposition 4. *Let* $\mathrm{Shv}_c(\mathrm{Pro}(\mathcal{C}))$ *denote the full subcategory of* $\mathrm{Shv}(\mathrm{Pro}(\mathcal{C}))$ *consisting of those sheaves* $\mathscr{F} : \mathrm{Pro}(\mathcal{C})^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ *which preserve filtered colimits. Then the restriction functor* $\mathscr{F} \mapsto \mathscr{F}|_{\mathcal{C}^{\mathrm{op}}}$ *induces an equivalence of categories* $\mathrm{Shv}_c(\mathrm{Pro}(\mathcal{C})) \to \mathrm{Shv}(\mathcal{C})$.

Proposition 4 is the starting point of a strategy for understanding the topos $\mathrm{Shv}(\mathcal{C})$: its objects can also be understood as sheaves on the larger coherent category $\mathrm{Pro}(\mathcal{C})$, satisfying a certain continuity condition. This is convenient because $\mathrm{Pro}(\mathcal{C})$ contains many useful objects that do not belong to $\mathcal{C}$:

Definition 5. Recall that a *model* of $\mathcal{C}$ is a morphism of coherent categories $M : \mathcal{C} \to \mathcal{S}\mathrm{et}$: that is, a functor which satisfies the following axioms:

(1) The functor M commutes with finite limits.

(2) The functor M carries effective epimorphisms in $\mathcal{C}$ to surjections of sets.

(3) The functor M preserves finite coproducts.

Let $\mathrm{Mod}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ spanned by the models of $\mathcal{C}$. By definition, $\mathrm{Pro}(\mathcal{C})$ is the opposite of the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et})$ spanned by those functors which satisfy condition (1). We can therefore identify $\mathrm{Mod}(\mathcal{C})^{\mathrm{op}}$ with a full subcategory of $\mathrm{Pro}(\mathcal{C})$. Note that objects of $\mathrm{Mod}(\mathcal{C})^{\mathrm{op}}$ very rarely belong to $\mathcal{C}$ itself (regarded as a full subcategory of $\mathrm{Pro}(\mathcal{C})$ via the Yoneda embedding.

We will say that an object $M \in \mathrm{Pro}(\mathcal{C})$ is *weakly projective* if it satisfies conditions (1) and (2). We let $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ denote the full subcategory of $\mathrm{Pro}(\mathcal{C})$ spanned by the weakly projective objects.

Example 6. Any model of $\mathcal{C}$ is weakly projective when viewed as an object of $\mathrm{Pro}(\mathcal{C})$. That is, we have inclusions

$$\mathrm{Mod}(\mathcal{C}) \subseteq \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})^{\mathrm{op}} \subseteq \mathrm{Pro}(\mathcal{C})^{\mathrm{op}} \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{S}\mathrm{et}).$$

Example 7. Suppose that $\mathcal{C}$ is the category of finite sets. Then every effective epimorphism in $\mathcal{C}$ admits a section, so condition (2) of Definition 5 is automatic: that is, we have $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}) = \mathrm{Pro}(\mathcal{C})$.

Remark 8. By definition, an object $X \in \mathrm{Pro}(\mathcal{C})$ is weakly projective if and only if, for every effective epimorphism $C \to D$ in $\mathcal{C}$, the map $\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X, C) \to \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X, D)$ is surjective: that is, every map from X to D factors through C. It follows that $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ is closed under (possibly infinite) coproducts in $\mathrm{Pro}(\mathcal{C})$.

Beware that the map $\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X, C) \to \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X, D)$ is generally *not* surjective if we assume only that $C \to D$ is an effective epimorphism in $\mathcal{C}$ (this is the motivation for the using the modifier "weakly" to describe the condition of Definition 4).

Remark 9. The full subcategory $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}) \subseteq \mathrm{Pro}(\mathcal{C})$ is closed under filtered inverse limits (since the collection of surjections in $\mathcal{S}\mathrm{et}$ is closed under filtered direct limits).

The following result allows us to "resolve" any object of $\mathrm{Pro}(\mathcal{C})$ by weakly projective objects:

Proposition 10. *For every object $X \in \mathrm{Pro}(\mathcal{C})$, there exists an effective epimorphism $\rho_X : \lambda(X) \to X$ in $\mathrm{Pro}(\mathcal{C})$ where $\lambda(X)$ is weakly projective. Moreover, we can arrange that $\lambda(X)$ is a functor of X, that ρ_X is a natural transformation of functors, and that the functor λ commutes with filtered limits.*

Proof. We use the small object argument of Quillen. Let $\{C_i \to D_i\}_{i \in I}$ be a set of representatives for all isomorphism classes of effective epimorphisms in $\mathcal{C}$. For each object $X \in \mathrm{Pro}(\mathcal{C})$, set

$$C(X) = \prod_{i \in I} \prod_{\eta \in \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X, D_i)} C_i \qquad D(X) = \prod_{i \in I} \prod_{\eta \in \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X, D_i)} D_i,$$

where both products are formed in the category $\mathrm{Pro}(\mathcal{C})$. We have a tautological map $X \to D(X)$; we define $\lambda_1(X) = C(X) \times_{D(X)} X$. Note that that there is a projection map $\lambda_1(X) \to X$ in $\mathrm{Pro}(\mathcal{C})$, which is easily seen to be an effective epimorphism. For $n > 1$, we define $\lambda_n(X)$ by the formula $\lambda_n(X) = \lambda_1(\lambda_{n-1}(X))$, so that we have an inverse system

$$\cdots \to \lambda_3(X) \to \lambda_2(X) \to \lambda_1(X) \to X.$$

Set $\lambda(X) = \varprojlim \lambda_n(X)$. Note that each map $f : \lambda(X) \to D_i$ factors through $f_n : \lambda_n(X) \to D_i$ for some $n \gg 0$. By construction, the composite map $\lambda_{n+1}(X) \to \lambda_n(X) \xrightarrow{f_n} D_i$ factors through C_i, so that $f : \lambda(X) \to D_i$ factors through C_i. It follows that $\lambda(X)$ is weakly projective. By inspection, the construction of $\lambda(X)$ (and the projection map $\lambda(X) \to X$) is functorial in X and commutes with filtered limits. □

We will say that a collection of morphisms $\{U_i \to X\}_{i \in I}$ in $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ is a *covering* if it is a covering in $\mathrm{Pro}(\mathcal{C})$: that is, if there is a finite subset $I_0 \subseteq I$ such that $\coprod_{i \in I_0} U_i \to X$ is an effective epimorphism in $\mathrm{Pro}(\mathcal{C})$ (note that in this case, $\coprod_{i \in I_0} U_i$ is also weakly projective). This determines a Grothendieck topology on the category $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$.

Warning 11. In Lecture 8, we defined the notion of a *Grothendieck topology* on a category $\mathcal{E}$ under the assumption that $\mathcal{E}$ admits finite limits. In general, the category $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ need not admit finite limits. In such cases, we must replace condition $(T1)$ appearing in Lecture 8 with the following:

$(T1')$ For every covering $\{U_i \to X\}$ in $\mathcal{E}$ and every morphism $Y \to X$ in $\mathcal{E}$, there exists a covering $\{V_j \to Y\}$ for which each of the maps $V_j \to Y \to X$ factors through some U_i.

We also need to revise the notion of sheaf. A functor $\mathscr{F} : \mathcal{E}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ is said to be a sheaf if, for every covering $\{U_i \to X\}$ in $\mathcal{E}$, the canonical map

$$\mathscr{F}(X) \to \varprojlim \mathscr{F}(U)$$

is a bijection, where the limit is taken over the sieve on X generated by the objects U_i (see Definition 13 of Lecture 9).

Example 12. Let $\mathcal{C}$ be the category of finite sets. Then $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}) = \mathrm{Pro}(\mathcal{C})$ can be identified with the category of Stone spaces. The preceding topology can be described as follows: a finite collection of maps of Stone spaces $\{Y_i \to X\}$ is a covering if and only if the induced map $\coprod Y_i \to X$ is surjective.

Proposition 13. *The construction $\mathscr{F} \mapsto \mathscr{F}|_{\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})^{\mathrm{op}}}$ induces an equivalence of categories $\mathrm{Shv}(\mathrm{Pro}(\mathcal{C})) \to \mathrm{Shv}(\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}))$. Moreover, a sheaf $\mathscr{F} : \mathrm{Pro}(\mathcal{C})^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ commutes with filtered colimits if and only if $\mathscr{F}|_{\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})^{\mathrm{op}}}$ commutes with filtered colimits.*

Proof. Let $\mathscr{F} \in \mathrm{Shv}(\mathrm{Pro}(\mathcal{C}))$. For each object $X \in \mathrm{Pro}(\mathcal{C})$, let $\lambda(X)$ be defined as in Proposition 11, and set $\mu(X) = \lambda(\lambda(X) \times_X \lambda(X))$. We then have an equalizer diagram

$$\mathscr{F}(X) \to \mathscr{F}(\lambda(X)) \rightrightarrows \mathscr{F}(\mu(X)),$$

so that we can functorially recover $\mathscr{F}(X)$ from the values of $\mathscr{F}$ on weakly projective objects. This gives an explicit left inverse to the restriction functor

$$\mathrm{Shv}(\mathrm{Pro}(\mathcal{C})) \to \mathrm{Shv}(\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})) \qquad \mathscr{F} \mapsto \mathscr{F}|_{\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})^{\mathrm{op}}};$$

we leave it to the reader to verify that it is a right inverse as well.

It is clear that if $\mathscr{F}$ commutes with filtered colimits, then so does the restriction $\mathscr{F}|_{\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})^{\mathrm{op}}}$. The converse follows from the formula

$$\mathscr{F}(X) = \mathrm{Eq}(\mathscr{F}(\lambda(X)) \rightrightarrows \mathscr{F}(\mu(X))),$$

since the constructions $X \mapsto \lambda(X)$ and $X \mapsto \mu(X)$ both preserve filtered inverse limits (as functors from $\mathrm{Pro}(\mathcal{C})$ to itself). □

Corollary 14. *Let* $\mathrm{Shv}_c(\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}))$ *be the full subcategory of* $\mathrm{Shv}(\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}))$ *spanned by those sheaves* $\mathscr{F} : \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ *which preserve filtered colimits. Then there is a canonical equvialence of categories* $\mathrm{Shv}(\mathcal{C}) \simeq \mathrm{Shv}_c(\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}))$.

Proof. Combine Propositions 14 and 4. □