Lecture 15: Spaces and Locales

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In the previous lecture, we introduced the notion of a *locale*. Every topological space X determines a locale $\mathcal{U}(X)$, given by the poset of open subsets of X. Note that if $f : X \to Y$ is a continuous map of topological spaces, then the formation of inverse images $f^{-1} : \mathcal{U}(Y) \to \mathcal{U}(X)$ preserves arbitrary unions (which are joins in $\mathcal{U}(X)$ and $\mathcal{U}(Y)$) and finite intersections (which are meets in $\mathcal{U}(X)$ and $\mathcal{U}(Y)$), and can therefore be regarded as a morphism of locales from $\mathcal{U}(X)$ to $\mathcal{U}(Y)$. Consequently, the construction

$$X \mapsto \mathcal{U}(X)$$

can be regarded as a functor from the category of topological spaces to the category of locales. In this lecture, we will show that this functor is not too far from being an equivalence of categories.

We now construct a functor in the opposite direction.

Definition 1. Let $\{0 < 1\}$ be the poset with two elements 0 and 1, which we can identify with the locale of open subsets of the one-point space $*$.

Let $\mathcal{U}$ be any locale. A *point* of $\mathcal{U}$ is a morphism of locales $\{0 < 1\} \to \mathcal{U}$, which is given by a map of posets $x^* : \mathcal{U} \to \{0 < 1\}$ which preserves finite meets and arbitrary joins. Note that such a map is determined by the subset $\mathcal{U}_x = \{U \in \mathcal{U} : x^*(U) = 1\} \subseteq \mathcal{U}$, which is required to satisfy the following conditions:

- (a) If $U \leq V$ and U belongs to $\mathcal{U}_x$, then V also belongs to $\mathcal{U}_0$. (This guarantees that the map x^* is order-preserving.)
- (b) If a join $\bigvee U_i$ belongs to $\mathcal{U}_x$, then some U_i belongs to $\mathcal{U}_x$ (this guarantees that the map x^* preserves joins).
- (c) The subset $\mathcal{U}_x$ is closed under finite meets (this guarantees that the map x^* preserves meets).

Note that conditions (a) and (b) are equivalent to the requirement that there exists an element $U(x) \in \mathcal{U}$ such that $\mathcal{U}_x = \{U \in \mathcal{U} : U \nleq U_x\}$; we can define $U(x)$ as the join $\bigvee_{U \notin \mathcal{U}_x} U$. In this case, we can restate (c) as follows:

- (c') The element $U(x) \in \mathcal{U}$ is *prime*: that is, it is not equal to the largest element $\mathbf{1} \in \mathcal{U}$ and, whenever $U(x) = V \wedge W$, we have either $U(x) = V$ or $U(x) = W$.

Remark 2. In the situation of Definition 1, it is useful to think of $\mathcal{U}_x$ as "the collection of $U \in \mathcal{U}$ which contain x" and $U(x)$ as "the largest element of $\mathcal{U}$ which does not contain x."

Definition 3. Let $\mathcal{U}$ be a locale. We let $\mathrm{Pt}(\mathcal{U})$ denote the collection of all points of $\mathcal{U}$, which we can identify either with the collection of subsets $\mathcal{U}_x \subseteq \mathcal{U}$ satisfying (a), (b), and (c), or with the collection of elements $U(x) \in \mathcal{U}$ satisfying (c').

For each $U \in \mathcal{U}$, let $\widetilde{U} \subseteq \mathrm{Pt}(\mathcal{U})$ be the collection of all points $x \in \mathrm{Pt}(\mathcal{U})$ such that $U \in \mathcal{U}_x$. Note that the construction $U \mapsto \widetilde{U}$ carries joins in $\mathcal{U}$ to unions of subsets of $\mathrm{Pt}(\mathcal{U})$ (by virtue of (b)) and finite meets in $\mathcal{U}$ to finite intersections of subsets of $\mathrm{Pt}(\mathcal{U})$ (by virtue of (c)). It follows that the collection of subsets of $\mathrm{Pt}(\mathcal{U})$ having the form $\widetilde{U}$ determines a topology on $\mathrm{Pt}(\mathcal{U})$. We will henceforth regard $\mathrm{Pt}(\mathcal{U})$ as equipped with this topology.

Now suppose that we are given a topological space X and a locale $\mathcal{U}$. Note that a map $f : X \to \mathrm{Pt}(\mathcal{U})$ is continuous if and only if, for each $U \in \mathcal{U}$, the set $f^{-1}(\widetilde{U}) = \{x \in X : U \in \mathcal{U}_{f(x)}\}$ is open in X. In this case, we the construction $U \mapsto f^{-1}\widetilde{U}$ determines a morphism of locales from $\mathcal{U}(X)$ to $\mathcal{U}$. Conversely, if $g^* : \mathcal{U} \to \mathcal{U}(X)$ is morphism of locales from $\mathcal{U}(X)$ to $\mathcal{U}$, then the construction

$$(x \in X) \mapsto \{U \in \mathcal{U} : x \in g^* U\}$$

determines a continuous map from X to $\mathrm{Pt}(\mathcal{U})$. These constructions are mutually inverse:

Proposition 4. *For any topological space X and any locale $\mathcal{U}$, we have a canonical bijection*

$$\mathrm{Hom}_{\mathrm{Top}}(X, \mathrm{Pt}(\mathcal{U})) \simeq \mathrm{Hom}_{Locales}(\mathcal{U}(X), \mathcal{U}) = \mathrm{Fun}^*(\mathcal{U}, \mathcal{U}(X)).$$

Consequently, we have a pair of adjoint functors

$$\text{Topological Spaces} \underset{\mathcal{U} \mapsto \mathrm{Pt}(\mathcal{U})}{\overset{X \mapsto \mathcal{U}(X)}{\rightleftarrows}} \text{Locales}\,.$$

Let X be a topological space. Then we can identify elements of $\mathrm{Pt}(\mathcal{U}(X))$ with open subsets $U \subseteq X$ which are *prime*, in the following sense:

($*$) $U \neq X$. Moreover, if U is an intersection $V_0 \cap V_1$ of open subsets $V_0, V_1 \subseteq X$, then either $U = V_0$ or $U = V_1$.

We can rephrase this as a condition on the *complement* $K = X - U$:

($*'$) The set K is nonempty. Moreover, if $K = K_0 \cup K_1$ written as a union of closed subsets of X, then either $K = K_0$ or $K = K_1$.

We say that a closed subset $K \subseteq X$ is *irreducible* if it satisfies condition ($*'$). We have proven:

Proposition 5. *Let X be a topological space. Then there is a canonical bijection*

$$\{ \text{ Irreducible closed subsets } K \subseteq X \text{ } \} \simeq \{ \text{ Points of } \mathcal{U}(X) \text{ } \}$$

This bijection carries an irreducible closed subset $K \subseteq X$ to the point x of $\mathcal{U}(X)$ given by $\mathcal{U}(X)_x = \{U \subseteq X : U \cap K \neq \emptyset\}$.

For any topological space X, we have a unit map $X \to \mathrm{Pt}(\mathcal{U}(X))$, which we can think of as a map of sets

$$X \to \{\text{Irreducible closed subsets } K \subseteq X\}.$$

Unwinding the definitions, we see that this map carries a point $x \in X$ to the subset $\overline{\{x\}} \subseteq X$.

Definition 6. Let X be a topological space. We say that X is *sober* if the map

$$X \to \{\text{Irreducible closed subsets } K \subseteq X\}.$$

is bijective. In other words, X is sober if every irreducible closed subset K of X has a unique "generic point" $x \in K$, characterized by the requirement that $K = \overline{\{x\}}$.

Proposition 7. *Let X be a topological space. Then X is sober if and only if the canonical map $X \to \mathrm{Pt}(\mathcal{U}(X))$ is a homeomorphism.*

Proof. The "only if" direction is obvious (since any homeomorphism is bijective). To prove the converse, we observe that every open set $U \subseteq X$ is the inverse image of an open subset of $\mathrm{Pt}(\mathcal{U}(X))$: namely, the open subset $\widetilde{U}$. □

We now give some examples of sober topological spaces.

Example 8. Let X be a Hausdorff space. Then a subset $K \subseteq X$ is closed and irreducible if and only if $K = \{x\}$ for some $x \in X$. In particular, every irreducible closed subset of X has a unique generic point, so that X is sober.

Example 9. Let X be the underlying topological space of a scheme. Then X is sober.

Proposition 10. *Let $\mathcal{U}$ be a locale. Then the topological space $\mathrm{Pt}(\mathcal{U})$ is sober.*

Proof. We wish to show that the unit map

$$u_{\mathrm{Pt}(\mathcal{U})} : \mathrm{Pt}(\mathcal{U}) \to \mathrm{Pt}(\mathcal{U}(\mathrm{Pt}(\mathcal{U})))$$

is bijective. Let $v_{\mathcal{U}} : \mathcal{U}(\mathrm{Pt}(\mathcal{U})) \to \mathcal{U}$ be the counit map (regarded as a map of locales), so that the composition

$$\mathrm{Pt}(\mathcal{U}) \xrightarrow{u_{\mathrm{Pt}(\mathcal{U})}} \mathrm{Pt}(\mathcal{U}(\mathrm{Pt}(\mathcal{U}))) \xrightarrow{\mathrm{Pt}(v_{\mathcal{U}})} \mathrm{Pt}(\mathcal{U})$$

is the identity. It follows immediately that $u_{\mathrm{Pt}(\mathcal{U})}$ is injective. To show surjectivity, it will suffice to show that the map of topological spaces $\mathrm{Pt}(v_{\mathcal{U}}) : \mathrm{Pt}(\mathcal{U}(\mathrm{Pt}(\mathcal{U}))) \to \mathrm{Pt}(\mathcal{U})$ is injective. In other words, it will suffice to show that a point x of the locale $\mathcal{U}(\mathrm{Pt}(\mathcal{U}))$ is determined by the set $\{U \in \mathcal{U} : x \in \widetilde{U}\}$. This is clear, because the map of posets

$$v_{\mathcal{U}}^{*} : \mathcal{U} \to \mathcal{U}(\mathrm{Pt}(\mathcal{U})) \qquad U \mapsto \widetilde{U}$$

is surjective. is surjective (by definition, every open subset of $\mathrm{Pt}(\mathcal{U})$ has the form $\widetilde{U}$ for some $U \in \mathcal{U}$). □

Corollary 11. *Let X be a topological space. Then X is sober if and only it is homeomorphic to $\mathrm{Pt}(\mathcal{U})$, for some locale $\mathcal{U}$.*

Definition 12. Let $\mathcal{U}$ be a locale. We say that $\mathcal{U}$ is *spatial* if the counit $\mathcal{U}(\mathrm{Pt}(\mathcal{U})) \to \mathcal{U}$ is an isomorphism of locales. In other words, $\mathcal{U}$ is spatial if the construction

$$(U \in \mathcal{U}) \mapsto (\widetilde{U} \subseteq \mathrm{Pt}(\mathcal{U}))$$

determines a bijection from $\mathcal{U}$ to the collection of open subsets of $\mathrm{Pt}(\mathcal{U})$.

Remark 13. Note that the construction

$$(U \in \mathcal{U}) \mapsto (\widetilde{U} \subseteq \mathrm{Pt}(\mathcal{U}))$$

is automatically surjective (by the definition of the topology on $\mathrm{Pt}(\mathcal{U})$). Consequently, $\mathcal{U}$ is spatial if and only if it is injective. In other words, $\mathcal{U}$ is spatial if it has *enough points*, in the sense that for any pair of distinct elements $U \neq V$ in $\mathcal{U}$, we can find a point x of $\mathcal{U}$ which distinguishes U from V (in the sense that $U \in \mathcal{U}_x$ and $V \notin \mathcal{U}_x$, or vice versa).

Proposition 14. *Let $\mathcal{U}$ be a locale. Then $\mathcal{U}$ is spatial if and only if it is isomorphic to $\mathcal{U}(X)$, for some topological space X.*

Proof. If $\mathcal{U}$ is spatial, then it is isomorphic to $\mathcal{U}(X)$ for $X = \mathrm{Pt}(\mathcal{U})$. Conversely, if X is any topological space, then the locale $\mathcal{U}(X)$ is spatial, since every pair of distinct open sets $U, V \subseteq X$ can be distinguished by a point of X (and therefore also by its image in $\mathrm{Pt}(\mathcal{U}(X))$). □

It follows that the adjunction

$$\text{Topological Spaces} \underset{\mathcal{U} \mapsto \mathrm{Pt}(\mathcal{U})}{\overset{X \mapsto \mathcal{U}(X)}{\rightleftarrows}} \text{Locales}\ .$$

restricts to an equivalence of categories

$$\{ \text{ Sober topological spaces } \} \simeq \{ \text{ Spatial locales } \}.$$

Since every locale of the form $\mathcal{U}(X)$ is spatial and every topological space of the form $\mathrm{Pt}(\mathcal{U})$ is sober, we also have the following:

Corollary 15. *The inclusion functor*

$$\{ \textit{ Spatial Locales } \} \hookrightarrow \{ \textit{ Locales } \}$$

admits a right adjoint, given by the construction $\mathcal{U} \mapsto \mathcal{U}(\mathrm{Pt}(\mathcal{U}))$. Concretely, this construction carries a locale $\mathcal{U}$ to the quotient $\mathcal{U} / \sim$, where $\sim$ denotes the equivalence relation

$$(U \sim V) \Leftrightarrow (\widetilde{U} = \widetilde{V})$$

Corollary 16. *The inclusion functor*

$$\{ \textit{ Sober topological spaces } \} \hookrightarrow \{ \textit{ Topological Spaces } \}$$

admits a left adjoint, given by the construction $X \mapsto \mathrm{Pt}(\mathcal{U}(X))$. Concretely, this left adjoint assigns to a topological space X the collection of all irreducible closed subsets of X, endowed with an appropriate topology.

Not every topological space is sober. For example, if X is a topological space with the trivial topology (defined by $\mathcal{U}(X) = \{\emptyset, X\}$), then X is sober if and only if it has cardinality ≤ 1. However, this can be viewed as a pathology of the notion of topological space: we are generally not very interested in topological spaces with the trivial topology. More generally, replacing a topological space X by its "soberification" $\mathrm{Pt}(\mathcal{U}(X))$ is in practice an inoffensive procedure.

Example 17. Let X be an algebraic variety over $\mathbf{C}$, which we identify with its set of $\mathbf{C}$-valued points. Then we can endow X with the Zariski topology. This space is usually not sober (unless X is 0-dimensional): its irreducible closed subsets are the subvarieties of X. The associated sober space $\mathrm{Pt}(\mathcal{U}(X))$ is the underlying topological space of X as a scheme (where every subvariety has a generic point).

By contrast, replacing a locale $\mathcal{U}$ by the associated spatial locale $\mathcal{U}(\mathrm{Pt}(\mathcal{U}))$ is often very destructive. In this class, we will meet some interesting and useful examples of locales which have no points at all.

Example 18 (Deligne)**.** Let $\mathcal{U}$ denote the collection of equivalence classes of measurable subsets of the interval $[0, 1]$, where two measurable subsets X and Y are considered to be equivalent if $X - Y$ and $Y - X$ have measure zero. Then $\mathcal{U}$ is locale (in fact, it is a complete Boolean algebra) for which the associated topological space $\mathrm{Pt}(\mathcal{U})$ is empty.