Lecture 13: Elimination of Imaginaries

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Let us now return to the discussion of coherent topoi from Lecture 11. Recall that, if $\mathcal{X}$ is a coherent topos, then we can identify $\mathcal{X}$ with $\mathrm{Shv}(\mathcal{X}_{\mathrm{coh}})$, where $\mathcal{X}_{\mathrm{coh}} \subseteq \mathcal{X}$ is the full subcategory of coherent objects. We now note a few closure properties of this subcategory.

Lemma 1. *Let $\mathcal{X}$ be a coherent topos. Then the collection of coherent objects is closed under finite coproducts.*

Proof. Let $\{X_i\}_{i \in I}$ be a collection of coherent objects of $\mathcal{X}$ indexed by a finite set I, having coproduct $X = \coprod_{i \in I} X_i$. Then X is quasi-compact; we claim that it is also quasi-separated. Choose quasi-compact objects $U, V \in \mathcal{X}$ with maps $U \to X \leftarrow V$. For each $i \in I$, set $U_i = U \times_X X_i$ and $V_i = V \times_X X_i$. Then $U \times_X V$ can be identified with the coproduct $\coprod_{i \in I} U_i \times_{X_i} V_i$. Since U_i and V_i are quasi-compact and X_i is quasi-separated, the fiber product $U_i \times_{X_i} V_i$ is also quasi-compact. It follows that $U \times_X V$ is quasi-compact. □

Lemma 2. *Let $\mathcal{X}$ be a coherent topos. Suppose that we are given an effective epimorphism $f : U \to X$ in $\mathcal{X}$. If U is coherent and the equivalence relation $U \times_X U$ is quasi-compact, then X is coherent.*

Proof. Since X is a quotient of U, it is quasi-compact. We will show that it is quasi-separated by verifying condition $(*)$ of Lecture 11. Suppose we are given a quasi-compact object Y and a pair of maps $g, g' : Y \to X$; we wish to show that the equalizer $\mathrm{Eq}(Y \rightrightarrows X)$ is quasi-compact. Choose an effective epimorphism $Y' \to (U \times U) \times_{X \times X} Y$, where Y' is quasi-compact. Then we have an effective epimorphism

$$\mathrm{Eq}(Y' \rightrightarrows X) \simeq Y' \times_Y \mathrm{Eq}(Y \rightrightarrows X) \to \mathrm{Eq}(Y \rightrightarrows X).$$

It will therefore suffice to show that $\mathrm{Eq}(Y' \rightrightarrows X)$ is quasi-compact. We may therefore replace Y by Y' and thereby reduce to the case where $g = f \circ \overline{g}$ and $g' = f \circ \overline{g}'$ for some pair of maps $\overline{g}, \overline{g}' : Y \to U$. In this case, we have a canonical isomorphism $\mathrm{Eq}(Y \rightrightarrows X) \simeq (Y \times_{U \times U} (U \times_X U))$. Since Y and $U \times_X U$ are quasi-compact and $U \times U$ is quasi-separated, it follows that $\mathrm{Eq}(Y \rightrightarrows X)$ is quasi-compact, as desired. □

Proposition 3. *Let $\mathcal{X}$ be a coherent topos and let $\mathcal{X}_{\mathrm{coh}} \subseteq \mathcal{X}$ be the full subcategory spanned by the coherent objects. Then $\mathcal{X}_{\mathrm{coh}}$ is an (essentially small) pretopos.*

Proof. We proved in Lecture 9 that $\mathcal{X}$ is a pretopos. In particular, it admits finite limits, finite coproducts, and every equivalence relation $R \subseteq U \times U$ can be obtained as the fiber product $U \times_X U$, for some effective epimorphism $U \to X$. We proved in Lecture 11 that the subcategory $\mathcal{X}_{\mathrm{coh}} \subseteq \mathcal{X}$ is closed under the formation of finite limits, and Lemmas 1 and 2 guarantee that it is also closed finite coproducts, and quotients by equivalence relations. From this it is easy to see that $\mathcal{X}_{\mathrm{coh}}$ is also a pretopos (check this as an exercise), and we saw in Lecture 11 that it is essentially small. □

Let $\mathcal{C}$ be a coherent category. Recall that $\mathcal{C}$ can be equipped with a Grothendieck topology, where a collection of morphisms $\{u_i : U_i \to X\}_{i \in I}$ is a covering if there exists a finite subset $I_0 \subseteq I$ such that $X = \bigvee_{i \in I_0} \mathrm{Im}(u_i)$. In Lecture 8, we saw that this Grothendieck topology is subcanonical: that is, the Yoneda embedding determines a functor $h : \mathcal{C} \to \mathrm{Shv}(\mathcal{C})$. Moreover, it is also finitary, so that $\mathrm{Shv}(\mathcal{C})$ is a coherent topos and the functor h takes values in the subcategory $\mathrm{Shv}(\mathcal{C})_{\mathrm{coh}} \subseteq \mathrm{Shv}(\mathcal{C})$ of coherent objects.

Exercise 4. Let $\mathcal{C}$ and $\mathcal{D}$ be coherent categories and let $f : \mathcal{C} \to \mathcal{D}$ be a functor which preserves finite limits. Show that the following conditions are equivalent:

(1) The functor f is a morphism of coherent categories: that is, it preserves effective epimorphisms and (finite) joins of subobjects.

(2) The functor f carries each covering $\{U_i \to X\}$ in $\mathcal{C}$ to a covering $\{f(U_i) \to f(X)\}_{i \in I}$ in $\mathcal{D}$.

Proposition 5. *Let $\mathcal{C}$ be a coherent category and let $h : \mathcal{C} \to \mathrm{Shv}(\mathcal{C})$ be the Yoneda embedding. Then:*

(1) *Let $\mathscr{G}$ be a coherent object of $\mathrm{Shv}(\mathcal{C})$ and let $\mathscr{F} \subseteq \mathscr{G}$ be a coherent subobject. If $\mathscr{G}$ belongs to the essential image of h, then so does $\mathscr{F}$.*

(2) *If $\mathcal{C}$ is a pretopos, then the Yoneda embedding h induces an equivalence of categories $\mathcal{C} \to \mathrm{Shv}(\mathcal{C})_{\mathrm{coh}}$.*

Proof. We first prove (1). Assume that $\mathscr{G} = h_X$ for some object $X \in \mathcal{C}$. Let $\mathscr{F} \subseteq \mathscr{G}$ be a coherent subobject, and choose a covering $\{h_{U_i} \to \mathscr{F}\}_{i \in I}$ in $\mathrm{Shv}(\mathcal{C})$. Since $\mathscr{F}$ is quasi-compact, we can assume that I is finite. Note that each of the maps $h_{U_i} \to \mathscr{F}$ can be identified with a map from h_{U_i} to h_X, and therefore (by Yoneda's lemma) arises from a map $u_i : U_i \to X$ in the category $\mathcal{C}$. Since the category $\mathcal{C}$ is coherent, we can form the join $X_0 = \bigvee_{i \in I} \mathrm{Im}(u_i)$. Since the functor h preserves images and joins of subobjects (Exercise 4), it follows that $\mathscr{F} \simeq h_{X_0}$ belongs to the essential image of h.

We now prove (2). The Yoneda embedding $h : \mathcal{C} \to \mathrm{Shv}(\mathcal{C})$ is a morphism of coherent categories (Exercise 4), and $\mathrm{Shv}(\mathcal{C})$ is a pretopos. If $\mathcal{C}$ is also a pretopos, then h preserves finite coproducts. Let $\mathscr{F} \in \mathrm{Shv}(\mathcal{C})$ be a coherent object, and choose a covering $\{h_{X_i} \to \mathscr{F}\}_{i \in I}$. Since $\mathscr{F}$ is quasi-compact, we can assume that I is finite. Setting $X = \coprod_i X_i$ (and noting that h preserves coproducts), we obtain an effective epimorphism $h_X \to \mathscr{F}$. Note that $h_X \times_{\mathscr{F}} h_X$ can be identified with a subobject of $h_{X \times X}$. Using (1), we can write $h_X \times_{\mathscr{F}} h_X = h_R$ for some subobject $R \subseteq X \times X$. Then R is an equivalence relation on X, and our assumption that $\mathcal{C}$ is a pretopos guarantees that we have $R = X \times_Y X$ for some effective epimorphism $X \to Y$. It then follows that $\mathscr{F} \simeq h_Y$ belongs to the essential image of h. □

It follows from Proposition 5 that the datum of a coherent topos $\mathcal{X}$ is equivalent to the datum of an essentially small pretopos $\mathcal{C}$: from an essentially small pretopos $\mathcal{C}$ we can construct a coherent topos $\mathrm{Shv}(\mathcal{C})$, and from a coherent topos $\mathcal{X}$ we can extract an essentially small pretopos $\mathcal{X}_{\mathrm{coh}}$; these processes are mutually inverse to one another. Beware, however, that the 2-category of pretopoi (with maps given by morphisms of coherent categories) is not quite equivalent to the 2-category of coherent topoi (with maps given by geometric morphisms): see Corollary 7 below.

Proposition 6. *Let $\mathcal{C}$ be a small coherent category and let $\mathcal{X}$ be a topos. Then composition with the Yoneda embedding $h : \mathcal{C} \to \mathrm{Shv}(\mathcal{C})$ induces a fully faithful embedding*

$$\mathrm{Fun}^*(\mathrm{Shv}(\mathcal{C}), \mathcal{X}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{X}),$$

whose essential image consists of those functors $f : \mathcal{C} \to \mathcal{X}$ which are morphisms of coherent categories.

Proof. By virtue of the main result of Lecture 12, it will suffice to show that a functor $f : \mathcal{C} \to \mathcal{X}$ is a morphism of coherent categories if and only if it preserves finite limits and carries coverings in $\mathcal{C}$ to coverings in $\mathcal{X}$. This is a special case of Exercise 4.

□

Corollary 7. *Let $\mathcal{C}$ be a small coherent category and let $\mathcal{D}$ be a small pretopos. Then the category $\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{D})$ of morphisms of coherent categories from $\mathcal{C}$ to $\mathcal{D}$ can be identified with the full subcategory of $\mathrm{Fun}^*(\mathrm{Shv}(\mathcal{C}), \mathrm{Shv}(\mathcal{D}))$ spanned by those geometric morphisms $f^* : \mathrm{Shv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{D})$ which carry coherent objects to coherent objects.*

Proof. Set $\mathrm{Shv}(\mathcal{D})$. Let us abuse notation by identifying $\mathcal{D}$ with the full subcategory $\mathrm{Shv}(\mathcal{D})_{\mathrm{coh}}$ of coherent objects of $\mathrm{Shv}(\mathcal{D})$ (Proposition 5). Then Proposition 6 supplies an equivalence $\mathrm{Fun}^*(\mathrm{Shv}(\mathcal{C}), \mathrm{Shv}(\mathcal{D})) \simeq \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathrm{Shv}(\mathcal{D}))$. It will therefore suffice to show that a morphism of coherent categories $f : \mathcal{C} \to \mathrm{Shv}(\mathcal{D})$ sends each object of $\mathcal{C}$ into $\mathrm{Shv}(\mathcal{D})_{\mathrm{coh}}$ if and only if the induced geometric morphism $F : \mathrm{Shv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{D})$ carries each coherent object of $\mathrm{Shv}(\mathcal{C})$ into $\mathrm{Shv}(\mathcal{D})_{\mathrm{coh}}$. The "if" direction is obvious; we leave the converse as an exercise. □

Construction 8. Let $\mathcal{C}$ be a small coherent category. We let $\mathcal{C}_{\mathrm{eq}}$ denote the full subcategory of $\mathrm{Shv}(\mathcal{C})$ spanned by the coherent objects. Note that the Yoneda embedding $h : \mathcal{C} \to \mathrm{Shv}(\mathcal{C})$ determines a morphism of coherent categories $h : \mathcal{C} \to \mathcal{C}_{\mathrm{eq}}$.

Proposition 9. *Let $\mathcal{C}$ be a small coherent category. Then the functor $h : \mathcal{C} \to \mathcal{C}_{\mathrm{eq}}$ exhibits $\mathcal{C}_{\mathrm{eq}}$ as a pretopos completion of $\mathcal{C}$, in the sense of Lecture 7.*

Proof. Let $\mathcal{D}$ be a pretopos; we wish to show that composition with h induces an equivalence $\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}_{\mathrm{eq}}, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{D})$. Writing $\mathcal{D}$ as a filtered union of small pretopoi, we can reduce to the case where $\mathcal{D}$ is essentially small. Using Proposition 5, we can reduce to the case where $\mathcal{D} = \mathcal{Y}_{\mathrm{coh}}$, where $\mathcal{Y}$ is a coherent topos.

Set $\mathcal{X} = \mathrm{Shv}(\mathcal{C})$. Then $\mathcal{X}$ is a coherent topos, and can therefore be identified with the category of sheaves $\mathrm{Shv}(\mathcal{X}_{\mathrm{coh}}) = \mathrm{Shv}(\mathcal{C}_{\mathrm{eq}})$. Let $\mathrm{Fun}'(\mathcal{X}, \mathcal{Y})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{X}, \mathcal{Y})$ spanned by those functors which preserve small colimits, finite limits, and coherent objects. We have restriction functors

$$\mathrm{Fun}'(\mathcal{X}, \mathcal{Y}) \to \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}_{\mathrm{eq}}, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{D}).$$

It follows from Corollary 7 that the left map and the composite map are equivalences of categories, so the right map is an equivalence of categories as well. □

We close this lecture with a (hopefully) illuminating example.

Definition 10. Let $\mathcal{C}$ be a category which admits finite limits. A *group object* of $\mathcal{C}$ is an object $G \in \mathcal{C}$ equipped with a map $m : G \times G \to G$ with the following property: for each object $C \in \mathcal{C}$, the induced multiplication

$$\mathrm{Hom}_{\mathcal{C}}(C, G) \times \mathrm{Hom}_{\mathcal{C}}(C, G) \simeq \mathrm{Hom}_{\mathcal{C}}(C, G \times G) \xrightarrow{m\circ} \mathrm{Hom}_{\mathcal{C}}(C, G)$$

endows $\mathrm{Hom}_{\mathcal{C}}(C, G)$ with the structure of a group. The collection of group objects of $\mathcal{C}$ forms a category, which we will denote by $\mathrm{Group}(\mathcal{C})$.

Example 11. In the case $\mathcal{C} = \mathcal{S}\mathrm{et}$, we will denote $\mathrm{Group}(\mathcal{C})$ simply by Group; this is the usual category of groups.

Remark 12. Let $\mathcal{C}$ be a category which admits finite limits and let G be a group object of $\mathcal{C}$. For every group $\Gamma \in \mathrm{Group}$, the construction

$$(C \in \mathcal{C}) \mapsto \mathrm{Hom}_{\mathrm{Group}}(\Gamma, \mathrm{Hom}_{\mathcal{C}}(C, G))$$

determines a functor from $\mathcal{C}^{\mathrm{op}}$ to the category of sets, which we will denote by G^{Γ}.

Note that if Γ is given by generators $\{x_i\}_{i \in I}$ and relations $\{u_j = v_j\}_{j \in J}$, then the presheaf G^{Γ} can be realized as an equalizer

$$G^{\Gamma} \to \prod_{i \in I} h_G \rightrightarrows \prod_{j \in J} h_G.$$

In particular, if Γ is finitely generated, then the presheaf G^{Γ} is representable by an object of $\mathcal{C}$; we will abuse notation by identifying this object with G^{Γ}.

Exercise 13. Let $\mathcal{C}$ be a category which admits finite limits and let $\mathrm{Group}_{\mathrm{fp}}$ denote the full subcategory of Group spanned by the finitely presented groups. Show that the construction

$$(G \in \mathrm{Group}(\mathcal{C})) \mapsto \{G^{\Gamma}\}_{\Gamma \in \mathrm{Group}_{\mathrm{fp}}}$$

induces a fully faithful embedding

$$\mathrm{Group}(\mathcal{C}) \to \mathrm{Fun}(\mathrm{Group}_{\mathrm{fp}}^{\mathrm{op}}, \mathcal{C}).$$

Moreover, the essential image of this embedding consists of those functors $\mathrm{Group}_{\mathrm{fp}}^{\mathrm{op}} \to \mathcal{C}$ which preserve finite limits.

Hint: to go the reverse direction, suppose we are given a functor $F : \mathrm{Group}_{\mathrm{fp}}^{\mathrm{op}} \to \mathcal{C}$ which preserves finite limits. Let $\langle x \rangle \simeq \mathbf{Z}$ be the free group on one generator and let $\langle x_0, x_1 \rangle$ be the free group on 2 generators. Set $G = F(\langle x \rangle)$, and let

$$m : G \times G \simeq F(\langle x_0 \rangle) \times F(\langle x_1 \rangle) \simeq F(\langle x_0, x_1 \rangle) \to F(\langle x \rangle) \simeq G$$

be the map obtained by applying F to the group homomorphism

$$\langle x \rangle \to \langle x_0, x_1 \rangle \qquad x \mapsto x_0 x_1.$$

Show that m exhibits G as a group object of $\mathcal{C}$, and that the construction $F \mapsto G$ is inverse to the construction $(G \in \mathrm{Group}(\mathcal{C})) \mapsto \{G^{\Gamma}\}_{\Gamma \in \mathrm{Group}_{\mathrm{fp}}}$.

Definition 14. Let $\mathcal{X} = \mathrm{Fun}(\mathrm{Group}_{\mathrm{fp}}, \mathcal{S}\mathrm{et})$ denote the category of presheaves on $\mathrm{Group}_{\mathrm{fp}}^{\mathrm{op}}$. We will refer to $\mathcal{X}$ as the *classifying topos of groups.*

For any topos $\mathcal{Y}$, let $\mathrm{Fun}^*(\mathcal{X}, \mathcal{Y})$ be the category of geometric morphisms from $\mathcal{Y}$ to $\mathcal{X}$. The main result of Lecture 12 shows that composition with the Yoneda embedding $h : \mathrm{Group}_{\mathrm{fp}}^{\mathrm{op}} \hookrightarrow \mathrm{Fun}(\mathrm{Group}_{\mathrm{fp}}, \mathcal{S}\mathrm{et}) = \mathcal{X}$ induces an equivalence of categories

$$\mathrm{Fun}^*(\mathcal{X}, \mathcal{Y}) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathrm{Group}_{\mathrm{fp}}^{\mathrm{op}}, \mathcal{Y}) \simeq \mathrm{Group}(\mathcal{Y}).$$

In particular, the category of geometric morphisms from $\mathcal{S}\mathrm{et}$ to $\mathcal{X}$ can be identified with the category $\mathrm{Group} = \mathrm{Group}(\mathcal{S}\mathrm{et})$ of groups.

Note that the topos $\mathcal{X}$ is coherent. Applying Proposition 6 to the pretopos $\mathcal{X}_{\mathrm{coh}}$, we obtain equivalences

$$\mathrm{Mod}(\mathcal{X}_{\mathrm{coh}}) = \mathrm{Fun}^{\mathrm{coh}}(\mathcal{X}_{\mathrm{coh}}, \mathcal{S}\mathrm{et}) \simeq \mathrm{Fun}^*(\mathcal{X}, \mathcal{S}\mathrm{et}) \simeq \mathrm{Group}(\mathcal{S}\mathrm{et}) = \mathrm{Group}\,.$$

That is, $\mathcal{X}_{\mathrm{coh}}$ is a pretopos whose models are groups.

Remark 15. In the above discussion, we can replace groups by other mathematical structures of a formally similar nature: abelian groups, monoids, rings, commutative rings, Lie algebras, $\cdots$.