Lecture 11: Coherent Topoi

February 22, 2018

Recall that if X is an object of a topos $\mathcal{X}$, we say that a collection of morphisms $\{f_i : U_i \to X\}_{i \in I}$ is a *covering* if the induced map $\amalg U_i \to X$ is an effective epimorphism.

Definition 1. Let $\mathcal{X}$ be a topos. We say that an object $X \in \mathcal{X}$ is *quasi-compact* if every covering of X has a finite subcovering. In other words, for every covering $\{U_i \to X\}_{i \in I}$, we can choose a finite subset $I_0 \subseteq I$ such that $\{U_i \to X\}_{i \in I_0}$ is also a covering.

Remark 2. Let X be an object of a topos $\mathcal{X}$. Then a collection of morphisms $\{f_i : U_i \to X\}$ is a covering if and only if the collection of subobjects $\{\mathrm{Im}(f_i) \subseteq X\}$ is a covering. Consequently, X is quasi-compact if and only if every covering of X by subobjects $\{U_i \subseteq X\}_{i \in I}$ admits a finite subcover.

Proposition 3. *Let $\mathcal{X}$ be a topos and let $f : X \to Y$ be an effective epimorphism in $\mathcal{X}$. If X is quasi-compact, then so is Y.*

Proof. Let $\{U_i \to Y\}_{i \in I}$ be a covering of Y. Then $\{U_i \times_Y X \to X\}_{i \in I}$ is a covering of X. Since X is quasi-compact, this cover admits a finite subcover $\{U_i \times_Y X \to X\}_{i \in I_0}$. Since f is an effective epimorphism, it follows that the collection of morphisms $\{U_i \times_Y X \to Y\}_{i \in I_0}$ is a covering of Y. In particular, the maps $\{U_i \to Y\}_{i \in I_0}$ give a covering of Y. □

Exercise 4. Let $\mathcal{X}$ be a topos and let $\{X_i\}_{i \in I}$ be a collection of objects indexed by a finite set I, having a coproduct $X = \amalg_{i \in I} X_i$. Show that X is quasi-compact if and only if each X_i is quasi-compact. In particular, the initial object of $\mathcal{X}$ is always quasi-compact.

Definition 5. Let $\mathcal{X}$ be a topos. We will say that an object $X \in \mathcal{X}$ is *quasi-separated* if, for every pair of morphisms $U \to X \leftarrow V$, where U and V are quasi-compact, the fiber product $U \times_X V$ is also quasi-compact.

Beware that the requirement of Definition 5 is sometimes satisfied for uninteresting reasons. For example, if $\mathcal{X} = \mathrm{Shv}(\mathbb{R}^n)$ is the category of sheaves on the Euclidean space $\mathbb{R}^n$, then the only quasi-compact object of $\mathcal{X}$ is the initial object. In this case, every object of $\mathcal{X}$ is quasi-separated. For Definition 5 to be meaningful, we need to ensure that there exists a good supply of quasi-compact objects.

Definition 6. Let $\mathcal{X}$ be a topos. We will say that $\mathcal{X}$ is *coherent* if there exists a collection of objects $\mathcal{U}$ satisfying the following conditions:

- The collection $\mathcal{U}$ generates $\mathcal{X}$: that is, every object $X \in \mathcal{X}$ admits a covering $\{U_i \to X\}$, where each U_i belongs to $\mathcal{U}$.
- The collection $\mathcal{U}$ is closed under finite products. In particular, it contains a final object of $\mathcal{X}$.
- Every object of $\mathcal{U}$ is quasi-compact and quasi-separated.

Remark 7. Let $\mathcal{X}$ be a coherent topos. Then the final object of $\mathcal{X}$ is quasi-separated. It follows that the collection of quasi-compact objects of $\mathcal{X}$ is closed under finite products.

We now describe a large class of examples of coherent topoi.

Definition 8. Let $\mathcal{C}$ be a category which admits finite limits. We say that a Grothendieck topology on $\mathcal{C}$ is *finitary* if, for every covering $\{U_i \to X\}_{i \in I}$ in $\mathcal{C}$, there exists a finite subset $I_0 \subseteq I$ such that $\{U_i \to X\}_{i \in I_0}$ is also a covering.

Proposition 9. *Let $\mathcal{C}$ be a small category which admits finite limits which is equipped with a finitary Grothendieck topology. Then the topos* $\mathrm{Shv}(\mathcal{C})$ *is coherent.*

Proof. Let $L : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathrm{Shv}(\mathcal{C})$ be the sheafification functor, and let $\mathcal{U}$ be the collection of all objects of the form Lh_C for $C \in \mathcal{C}$. We claim that $\mathcal{U}$ satisfies the requirements of Definition 6. We saw in the previous lecture that $\mathcal{U}$ generates $\mathrm{Shv}(\mathcal{C})$. Moreover, since the functor L preserves finite limits, the construction $C \mapsto Lh_C$ preserves finite limits. It follows that $\mathcal{U}$ is closed under finite products. It will therefore suffice to show that for each object $C \in \mathcal{C}$, the sheaf Lh_C is quasi-compact and quasi-separated. We first verify quasi-compactness. Choose a covering $\{\mathscr{F}_i \to Lh_C\}_{i \in I}$. Note that the identity map $\mathrm{id}_C : C \to C$ determines a section $s \in (Lh_C)(C)$. It follows that there exists a covering $\{C_j \to C\}_{j \in J}$ in the category $\mathcal{C}$ such that, for each $j \in J$, the image $s_j \in (Lh_C)(C_j)$ of s can be lifted to an element $\widetilde{s}_j \in \mathscr{F}_{i_j}(C_j)$ for some $i_j \in I$. Since the topology on $\mathcal{C}$ is finitary, we may assume without loss of generality that J is finite. Setting $I_0 = \{i_j\}_{j \in J} \subseteq I$, we deduce that $\{\mathscr{F}_i \to Lh_C\}_{i \in I_0}$ is a finite subcover of $\{\mathscr{F}_i \to Lh_C\}_{i \in I}$.

We now argue that each Lh_C is quasi-separated. Choose quasi-compact objects $\mathscr{F}, \mathscr{G} \in \mathrm{Shv}(\mathcal{C})$ equipped with maps $\mathscr{F} \to Lh_C \leftarrow \mathscr{G}$; we wish to show that the fiber product $\mathscr{F} \times_{Lh_C} \mathscr{G}$ is quasi-compact. Note that $\mathscr{F}$ admits a covering $\{\mathscr{F}_i \to \mathscr{F}\}_{i \in I}$, where each $\mathscr{F}_i$ belongs to $\mathcal{U}$. Since $\mathscr{F}$ is quasi-compact, we may assume that I is finite. Then $\{\mathscr{F}_i \times_{Lh_C} \mathscr{G}\}_{i \in I}$ is a finite covering of $\mathscr{F} \times_{Lh_C} \mathscr{G}$. It will therefore suffice to show that each $\mathscr{F}_i \times_{Lh_C} \mathscr{G}$ is quasi-compact. Replacing $\mathscr{F}$ by $\mathscr{F}_i$, we are reduced to the case where $\mathscr{F}$ has the form Lh_D for some object $D \in \mathcal{C}$. In this case, the map $\mathscr{F} \to Lh_C$ can be identified with an element of $(Lh_C)(D)$. Passing to a covering of D (which we can also replace by a finite subcover), we may assume that this element lies in the image of the map $h_C(D) \to (Lh_C)(D)$. In other words, we may assume that the map $\mathscr{F} \to Lh_C$ arises from applying the functor $Lh_\bullet$ to a morphism $D \to C$ in the category $\mathcal{C}$. Similarly, we may assume that the map $\mathscr{G} \to Lh_C$ arises by applying $Lh_\bullet$ to a morphism $E \to C$ in $\mathcal{C}$. In this case, the fiber product $\mathscr{F} \times_{Lh_C} \mathscr{G}$ can be identified with $Lh_{D \times_C E}$, which is quasi-compact by the first part of the argument. □

Our next goal is to show that every coherent topos $\mathcal{X}$ arises from the construction of Proposition 9 in a canonical (though not unique) fashion.

Definition 10. Let $\mathcal{X}$ be a coherent topos. We will say that an object $X \in \mathcal{X}$ is *coherent* if it is quasi-compact and quasi-separated.

It will be convenient to employ another characterization of the class of quasi-separated objects.

Lemma 11. *Let $\mathcal{X}$ be a coherent topos. Then an object $X \in \mathcal{C}$ is quasi-separated if and only if it satisfies the following condition:*

- $(*)$ *For every quasi-compact object $U \in \mathcal{X}$ and every pair of morphisms $f, g : U \to X$, the equalizer* $\mathrm{Eq}(U \rightrightarrows X)$ *is quasi-compact.*

Proof. Suppose first that X is quasi-separated, and that we are given a pair of morphisms $f, g : U \to X$; we wish to show that the equalizer $\mathrm{Eq}(U \rightrightarrows X)$ is quasi-compact. Let $\mathcal{U}$ be as in Definition 6, and choose a covering $\{U_i \to U\}_{i \in I}$ where each U_i belongs to $\mathcal{U}$. Since U is quasi-compact, we can assume that this covering is finite. Then f and g induce maps $f_i, g_i : U_i \to X$, having an equalizer $\mathrm{Eq}(U_i \rightrightarrows X) \simeq \mathrm{Eq}(U \rightrightarrows X) \times_U U_i$. It follows that $\mathrm{Eq}(U \rightrightarrows X)$ admits a finite covering by the objects $\mathrm{Eq}(U_i \rightrightarrows X)$. It will therefore suffice to show that each $\mathrm{Eq}(U_i \rightrightarrows X)$ is quasi-compact. In other words, we may replace U by U_i and thereby reduce to the case where $U \in \mathcal{U}$.

Unwinding the definitions, we see that the equalizer $\mathrm{Eq}(U \rightrightarrows X)$ can be identified with the fiber product

$$(U \times_X U) \times_{U \times U} U.$$

Note that the product $U \times U$ belongs to $\mathcal{U}$, and is therefore quasi-separated. Since U and and $U \times_X U$ are quasi-compact (the second by virtue of our assumption that X is quasi-separated), it follows that $\mathrm{Eq}(U \rightrightarrows X)$ is also quasi-compact.

We now prove the converse. Assume that condition $(*)$ is satisfied; we wish to prove that X is quasi-separated. Choose quasi-compact objects $U, V \in \mathcal{X}$ equipped with maps $U \to X \leftarrow V$; we wish to show that the fiber product $U \times_X V$ is quasi-compact. Unwinding the definitions, we can identify $U \times_X V$ with the equalizer of a diagram $(U \times V) \rightrightarrows X$. The desired result now follows from $(*)$, since $U \times V$ is a compact object of $\mathcal{X}$ (Remark 7). □

Lemma 12. *Let $\mathcal{X}$ be a coherent topos. Then the collection of quasi-separated objects of $\mathcal{X}$ is closed under finite products.*

Proof. Let X and Y be quasi-separated objects of $\mathcal{X}$. Suppose we are given a quasi-compact object $U \in \mathcal{X}$ and a pair of maps $f, g : U \to X \times Y$. Since X satisfies condition $(*)$ of Lemma 11, we deduce that the equalizer $\mathrm{Eq}(U \rightrightarrows X)$ is a quasi-compact subobject of U. Note that the equalizer $\mathrm{Eq}(U \rightrightarrows X \times Y)$ can be identified with the equalizer of a diagram

$$\mathrm{Eq}(U \rightrightarrows X) \rightrightarrows Y,$$

and is therefore also quasi-compact (since Y satisfies the condition $(*)$. □

Lemma 13. *Let $\mathcal{X}$ be a coherent topos. Then the collection of coherent objects of $\mathcal{X}$ is closed under finite limits.*

Proof. It follows immediately from the definition that the final object of $\mathcal{X}$ is coherent. It will therefore suffice to show that the collection of coherent objects is closed under fiber products. Suppose we are given maps $X \to Y \leftarrow Z$, where X, Y, and Z are coherent. Then X and Z are quasi-compact and Y is quasi-separated, so the fiber product $X \times_Y Z$ is quasi-compact. Moreover, $X \times_Y Z$ is a subobject of the product $X \times Z$, which is quasi-separated by Lemma 12. It follows that $X \times_Y Z$ is also quasi-separated. □

Lemma 14. *Let $\mathcal{X}$ be any topos and let $X \in \mathcal{X}$ be an object. Then* $\mathrm{Sub}(X)$ *is a set.*

Remark 15. Since a topos $\mathcal{X}$ is usually a large category, Lemma 14 is not automatic. The collection of isomorphism classes of objects of $\mathcal{X}$ will almost always be a proper class. However, Lemma 14 asserts that the collection of isomorphism classes of *subobjects* of a fixed object X is bounded in size.

Proof of Lemma 14. Let $\mathcal{U}$ be a set of generators for $\mathcal{X}$. Let $\mathrm{Sub}_0(X)$ denote the collection of all subobjects of X of the form $\mathrm{Im}(f)$, where $f : U \to X$ is a morphism with $U \in \mathcal{U}$. Since $\mathcal{U}$ is a set and $\mathrm{Hom}_{\mathcal{X}}(U, X)$ is a set for each $U \in \mathcal{U}$, the collection $\mathrm{Sub}_0(X)$ is a set. Note that every subobject $X_0 \subseteq X$ admits a covering $\{U_i \to X_0\}$, where each U_i belongs to $\mathcal{U}$. It follows that, as an element of $\mathrm{Sub}(X)$, we can identify X_0 with the least upper bound of $\{Y \in \mathrm{Sub}_0(X) : Y \subseteq X_0\}$. In particular, the construction

$$(X_0 \in \mathrm{Sub}(X)) \mapsto (\{Y \in \mathrm{Sub}_0(X) : Y \subseteq X_0\} \subseteq \mathrm{Sub}_0(X))$$

determines a monomorphism from $\mathrm{Sub}(X)$ to the power set of $\mathrm{Sub}_0(X)$, so that $\mathrm{Sub}(X)$ is also a set. □

Lemma 16. *Let $\mathcal{X}$ be a topos and let $\mathcal{X}_0 \subseteq \mathcal{X}$ be the full subcategory of $\mathcal{X}$ spanned by the quasi-compact objects. Then $\mathcal{X}_0$ is essentially small (that is, it is equivalent to a small category).*

Proof. Let $\mathcal{U}$ be a set of generators for $\mathcal{X}$. Enlarging $\mathcal{U}$ if necessary, we may assume that $\mathcal{U}$ is closed under finite coproducts. For each object $X \in \mathcal{X}$, we can choose a covering $\{U_i \to X\}$, where each U_i belongs to U. If X is quasi-compact, this covering admits a finite subcover. We can therefore choose a single object $U \in \mathcal{U}$ and an effective epimorphism $U \to X$. It follows that X can be identified with the coequalizer of a diagram $R \rightrightarrows U$, where $R = U \times_X U \subseteq U \times U$. Note that for each $U \in \mathcal{U}$, there is a bounded number of possibilities for what the equivalence relation R can be (since $\mathrm{Sub}(U \times U)$ is a set, by virtue of Lemma 14). □

Theorem 17. *Let $\mathcal{X}$ be a coherent topos and let $\mathcal{X}_{\mathrm{coh}} \subseteq \mathcal{X}$ be the full subcategory spanned by the coherent objects. Then:*

(a) *The category $\mathcal{X}_{\mathrm{coh}}$ admits a finitary Grothendieck topology, where a collection of morphisms $\{U_i \to X\}$ in $\mathcal{X}_{\mathrm{coh}}$ is a covering if and only if it is a covering in X.*

(b) *For each $X \in \mathcal{X}$, let $h_X : \mathcal{X}_{\mathrm{coh}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ denote the functor represented by X (given by $h_X(Y) = \mathrm{Hom}_{\mathcal{X}}(Y, X)$). Then the construction $X \mapsto h_X$ induces an equivalence of categories $\mathcal{X} \simeq \mathrm{Shv}(\mathcal{X}_{\mathrm{coh}})$.*

Proof. Since $\mathcal{X}_{\mathrm{coh}}$ is an essentially small subcategory of $\mathcal{X}$ (Lemma 16) which is closed under finite limits (Lemma 13) and generates $\mathcal{X}$ (Definition 6), assertions (a) and (b) follow from Giraud's theorem (see Lecture 10). Note that the topology on $\mathcal{X}_{\mathrm{coh}}$ is finitary because every object of $\mathcal{X}_{\mathrm{coh}}$ is quasi-compact. □