Math 261y: von Neumann Algebras (Lecture 9)

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Let us begin this lecture by continuing the analysis of ideals in von Neumann algebras. Let A be a von Neumann algebra and let $I \subseteq A \subseteq B(V)$ be an ultraweakly closed $*$-ideal. We saw in the last lecture that I can be regarded as a von Neumann algebra acting on the Hilbert space $\overline{IV} \subseteq V$. In particular, the identity element of I corresponds, in $B(V)$, to the orthogonal projection e onto the closed subspace $\overline{IV} \subseteq V$. Since I is a left ideal, we have $AI = I$, so that $AIV \subseteq IV$. It follows that the subspace $\overline{IV} \subseteq V$ is A-invariant, so that the projection e also belongs to the commutant A'. We have proven the following:

Proposition 1. *Let A be a von Neumann algebra and let $I \subseteq A$ be a $*$-ideal which is ultraweakly closed. Then there exists a central Hermitian element $e \in A$ such that $e^2 = e$ and $I = eA = Ae = eAe$. It follows that A decomposes as a product $I \times J$, where $J = (1-e)A = A(1-e) = (1-e)A(1-e)$.*

Now suppose that A is a nonunital C^*-algebra. We can always enlarge A to a unital C^*-algebra $\widetilde{A}$ by adding a unit. We then have an exact sequence (of nonunital $*$-algebra homomorphisms)

$$0 \to A \to \widetilde{A} \xrightarrow{\phi} \mathbf{C} \to 0.$$

This gives an exact sequence of Banach spaces

$$0 \to A^{\vee\vee} \to \widetilde{A}^{\vee\vee} \to \mathbf{C} \to 0.$$

We may therefore identify $A^{\vee\vee}$ with the kernel of the von Neumann algebra homomorphism $\psi : E(\widetilde{A}) \simeq \widetilde{A}^{\vee\vee} \to \mathbf{C}$ determined by ϕ. The analysis of Example **??** implies that this homomorphism determines von Neumann algebra isomorphism $E(\widetilde{A}) \simeq \mathbf{C} \times I$, where $I = \ker(\psi)$. As a Banach space, I is isomorphic to $A^{\vee\vee}$.

We can summarize the situation as follows. The category of representations of A as a nonunital C^*-algebra is equivalent to the category of representations of $\widetilde{A}$ as a unital C^*-algebra, which is in turn equivalent to the category of representations of the von Neumann algebra $E(\widetilde{A}) \simeq A^{\vee\vee} \times \mathbf{C}$. The splitting reflects the fact that every nonunital representation V of A admits a canonical decomposition $V \simeq \overline{AV} \oplus V_0$, where $\overline{AV}$ is a nondegenerate representation of A and V_0 is a trivial representation of A (that is, every element of A annihilates V_0). We have proven:

Theorem 2. *Let A be a nonunital C^*-algebra. Then the double dual $A^{\vee\vee}$ admits the structure of a von Neumann algebra. Moreover, there is a (nonunital) $*$-algebra homomorphism $A \to A^{\vee\vee}$ which determines an equivalence from the category of von Neumann algebra representations of $A^{\vee\vee}$ to the category of nondegenerate representations of A.*

Example 3. Let V be a Hilbert space and let $K(V) \subseteq B(V)$ denote the space of compact operators on V. Then $K(V)$ is a $*$-ideal in $B(V)$ which is closed in the norm topology. It is therefore a nonunital C^*-algebra (which has a unit if and only if V is finite dimensional). The trace pairing

$$B^{\mathrm{tc}}(V) \times B(V) \to \mathbf{C}$$

$$(f,g) \mapsto \mathrm{tr}(fg)$$

restricts to a pairing $B^{\mathrm{tc}}(V) \times K(V) \to \mathbf{C}$, which determines a map $B^{\mathrm{tc}}(V) \to K(V)^{\vee}$. Since $K(V)$ is ultrastrongly dense in $B(V)$ this map is actually an isometry onto its image. With more effort, one can show that it is surjective: that is, we have isomorphisms

$$B^{\mathrm{tc}}(V) \simeq K(V)^{\vee} \qquad B(V) \simeq B^{\mathrm{tc}}(V)^{\vee}.$$

Taken together, we get an isomorphism $B(V) \simeq K(V)^{\vee\vee}$, which exhibits $B(V)$ as the envelope of the (nonunital) C^*-algebra $B(V)$.

We will need a generalization of Proposition 1, which describes ultraweakly closed left and right ideals in a von Neumann algebra A. First, we need a bit of a digression. Suppose we are given a von Neumann algebra $A \subseteq B(V)$ containing two operators f and g with $f^*f = g^*g$. Then, for each vector $v \in V$, we have

$$(fv, fv) = (f^*fv, v) = (g^*gv) = (gv, gv).$$

It follows that there is a well-defined, isometric map $u_0 : fV \to gV$, given by $u(fv) = gv$. This map extends to an isometry $\overline{fV} \to \overline{gV}$. It follows that there exists a unique map $u : V \to V$ which coincides with u_0 on fV and vanishes on $(fV)^{\perp}$. The map u is a *partial isometry*: that is, it factors as a composition

$$V \to \overline{fV} \to \overline{gV},$$

where the first map is orthogonal projection onto a subspace of V and the second map is an isometry. This is equivalent to the assertion that uu^* and u^*u are both projection operators on V (exercise).

Suppose that $T \in B(V)$ is an operator belonging to the commutant A'. If $v \in (fV)^{\perp}$, then

$$(fw, Tv) = (T^*fw, v) = (fT^*w, v) = 0$$

for all $w \in V$, so that $Tv \in (fV)^{\perp}$. It follows that

$$u(Tv) = 0 = Tu(v).$$

We also have

$$uTf(v) = ufT(v) = gT(v) = Tg(v) = Tuf(v).$$

It follows that $Tu = uT$. Since this is true for all $T \in A'$, we conclude that $u \in A'' = A$. We have proven:

Proposition 4. *Let A be a von Neumann algebra containing elements f and g with $f^*f = g^*g$. Then A contains a partial isometry u satisfying $uf = g$.*

Corollary 5 (Polar Decomposition)**.** *Let A be a von Neumann algebra containing an element f. Then f admits a decomposition $f = u|f|$, where u is a partial isometry and $|f|$ denotes the unique positive square root of f^*f.*

Corollary 6. *Let $A \subseteq B(V)$ be a von Neumann algebra and let I be an ultraweakly closed left ideal. Then $I = Ae$ for some projection e.*

Proof. Consider the intersection $A_0 = I \cap I^*$. If $x, y \in A_0$, then $xy \in xI \subseteq I$ and $xy \in I^*y \subseteq I^*$, so that $xy \in A_0$. It follows that A_0 is a nonunital $*$-subalgebra of $B(V)$, which is closed in the ultraweak topology. We can therefore write V as an orthogonal direct sum $V_0 \oplus V_1$, where $A_0V_0 = 0$ and V_1 is a nondegenerate representation of A_0. Let $e \in B(V)$ denote the operator given by orthogonal projection onto V_1; it follows from the nonunital version of von Neumann's theorem that $e \in A_0$. In particular, we deduce that $e \in I$ so that $Ae \subseteq I$. We claim that equality holds. To prove this, consider an arbitrary element $x \in I$. Then x^*x belongs to $I \cap I^* = A_0$. Since A_0 is a C^*-algebra (in fact a von Neumann algebra) in its own right, x^*x has a unique positive square root in A_0, which we will denote by $|x|$. Then $|x| = |x|e$. The polar decomposition of x gives $x = u|x|$ for some partial isometry $u \in A$. Then $x = u|x| = u|x|e \in Ae$, as desired. □

The projection e appearing in Corollary 6 is uniquely determined by the ideal I. To see this, let us use the notation e_W to denote orthogonal projection onto a closed subspace $W \subseteq V$. If A contains e_W, then Ae_W is an ultraweakly closed left ideal in A (it is the kernel of the map given by right multiplication by $1 - e_W = e_{W^\perp}$). Note that a projection $e_{W'}$ belongs to Ae_W if and only if $e_{W'} = e_{W'}e_W$: that is, if and only if $W' \subseteq W$. We may therefore characterize e_W as the "largest" projection which belongs to the ideal Ae_W. We have proven the following.

Corollary 7. *Let $A \subseteq B(V)$ be a von Neumann algebra. There is an order-preserving, one-to-one correspondence between ultraweakly closed left ideals of A and closed subspaces $W \subseteq V$ such that $e_W \in A$. The correspondence is given by $W \mapsto Ae_W$.*

Corollary 8. *Let $A \subseteq B(V)$ be a von Neumann algebra and let $I = Ae_W$ be an ultraweakly closed left ideal in A. The following conditions are equivalent:*

(1) *The ideal I is a right ideal.*

(2) *The projection e_W belongs to the center of A.*

(3) *The ideal I is a $*$-ideal.*

Proof. We have already seen that (3) $\Rightarrow$ (2). If e_W is central then $Ae_W = e_W A$ is a right ideal, so that (2) $\Rightarrow$ (1). If I is a right ideal, then $I^* = e_W A \subseteq I$, so that I is a $*$-ideal; this proves (1) $\Rightarrow$ (3). □

Theorem 9. *Let A be a C^*-algebra. Suppose there exists a Banach space M and a Banach space isomorphism $A \simeq M^\vee$. Assume further:*

($*$) *For each $a \in A$, left and right multiplication by a are continuous with respect to the weak $*$-topology.*

Then A is isomorphic to a von Neumann algebra.

Remark 10. In the statement of Theorem 9, assumption ($*$) is actually unnecessary: just knowing that A admits a Banach space predual guarantees that A is isomorphic to a von Neumann algebra.

Proof. For every continuous linear map $\phi : A \to M^\vee$, there is an adjoint map $\phi' : M \to A^\vee$, which dualizes to give a map of Banach spaces $\hat{\phi} : A^{\vee\vee} \to M^\vee$. The map $\hat{\phi}$ is continuous with respect to the weak $*$-topologies on $A^{\vee\vee}$ and $M^\vee$, respectively, and fits into a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & M^\vee \\ & \searrow \qquad \nearrow_{\hat{\phi}} & \\ & A^{\vee\vee}. & \end{array}$$

Moreover, $\hat{\phi}$ is uniquely determined by these properties (since A is dense in $A^{\vee\vee}$ with respect to the weak $*$-topology).

Let us identify A with $M^\vee$ and take ϕ to be the identity map. We then obtain a map of Banach spaces $r : A^{\vee\vee} \to A$, which is the identity on A. Let us denote the kernel of r by $K \subseteq A^{\vee\vee}$. We have seen that $A^{\vee\vee}$ has the structure of a von Neumann algebra, and that the weak $*$-topology on $A^{\vee\vee}$ coincides with the ultraweak topology. It follows that $K \subseteq A^{\vee\vee}$ is ultraweakly closed.

Let $m_a : A \to A$ be the map given by left multiplication by an element $a \in A$, and let $\widehat{m}_a : A^{\vee\vee} \to A^{\vee\vee}$ be given by left multiplication by the image of A. Consider the diagram

$$\begin{CD} A^{\vee\vee} @>{r}>> A \\ @V{\widehat{m}_a}VV @VV{m_a}V \\ A^{\vee\vee} @>{r}>> A. \end{CD}$$

Using assumption $(*)$, we see that all of the maps appearing in this diagram are weak $*$-continuous. The diagram commutes when restricted to the image of A in $A^{\vee\vee}$. Since this image is weak $*$-dense, we see that the diagram commutes. That is, r commutes with left multiplication by $a \in A$. It follows that for $b \in K = \ker(r)$, we have $ab \in K$. The function $a \mapsto ab$ is ultraweakly continuous, and K is ultraweakly closed. It follows that $ab \in K$ for all $a \in A^{\vee\vee}$: that is, $Ab \in K$. Since $b \in K$ was arbitrary, we conclude that K is a left ideal in $A^{\vee\vee}$. The same argument proves that K is a right ideal in $A^{\vee\vee}$, and therefore an ultraweakly closed $*$-ideal in $A^{\vee\vee}$ (Corollary 8). It follows that the quotient $A^{\vee\vee}/K$ inherits the structure of a von Neumann algebra (it is actually a direct factor of $A^{\vee\vee}$). We have a map of $*$-algebras

$$A \to A^{\vee\vee} \to A^{\vee\vee}/K$$

which is an isomorphism at the level of vector spaces, and therefore a C^*-algebra isomorphism. □