Math 261y: von Neumann Algebras (Lecture 8)

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Let $\mathcal{C}$ denote the category of C^*-algebras (in which the morphisms are homomorphisms of $*$-algebras) and let $\mathcal{D}$ denote the category of von Neumann algebras (in which the morphisms are ultraweakly continuous $*$-algebra homomorphisms). There is an evident forgetful functor

$$\mathcal{D} \to \mathcal{C},$$

which assigns to each von Neumann algebra A its underlying C^*-algebra. In this lecture, we will construct a *left adjoint* to this functor. In other words, we will show that for every C^*-algebra A, we can construct a von Neumann algebra $E(A)$ equipped with a $*$-algebra homomorphism $\rho : A \to E(A)$ having the following universal property: for every von Neumann algebra A', every $*$-algebra homomorphism $\phi : A \to A'$ admits a unique factorization

$$A \to E(A) \xrightarrow{\psi} A',$$

where ψ is an ultraweakly continuous $*$-algebra homomorphism. In this case, the von Neumann algebra $E(A)$ is determined uniquely up to equivalence by A; we will refer to it as the *von Neumann algebra envelope* of A.

To verify that a map $\rho : A \to E(A)$ exhibits $E(A)$ as a von Neumann algebra envelope of A, it will suffice to verify the following pair of properties:

(a) The image of ρ is ultraweakly dense in $E(A)$.

(b) Every $*$-algebra homomorphism $A \to B(V)$ extends to an ultraweakly continous $*$-algebra homomorphism $E(A) \to B(V)$.

Indeed, suppose that (a) and (b) are satisfied, and that we are given a $*$-algebra homomorphism $\phi : A \to A'$, where $A' \subseteq B(V)$ is a von Neumann algebra. It follows from (b) that ϕ extends to an ultraweakly continuous $*$-algebra homomorphism $\psi : E(A) \to B(V)$. Since $A' \subseteq B(V)$ is closed in the ultraweak topology, we conclude that $\psi^{-1}A' \subseteq E(A)$ is an ultraweakly closed subset containing $\rho(A)$. It follows from (a) that $\psi^{-1}A' = E(A)$, so that we can regard ψ as a map from $E(A)$ to A'. This proves the existence of the desired extension; the uniqueness follows from (a) by a continuity argument.

To construct $E(A)$, let $S(A)$ denote the collection of all states of A. For each state $\mu : A \to \mathbf{C}$, we let V_μ be the associated representation of A: that is, the completion of A with respect to the inner product $\langle a, b\rangle = \mu(b^* a)$. Let V denote the Hilbert space direct sum $\bigoplus_{\mu \in S(A)} V_\mu$, and let $E(A)$ denote the closure of A in $B(V)$ in the ultraweak topology. Then $E(A) \subseteq B(V)$ is a von Neumann algebra, and there is an evident map $A \to E(A)$ whose image is ultraweakly dense. To verify (b), we must show that every representation W of A extends to a von Neumann algebra representation of $E(A)$. We saw in the last lecture that every representation of A can be obtained as a direct sum of cyclic representations; we may therefore assume without loss of generality that W is cyclic. Then W (if nonzero) has the form V_μ for some state μ, in which case the desired result is obvious.

Our next goal is to describe the envelope $E(A)$ more explicitly. Since every C^*-algebra A admits a faithful representation on a Hilbert space, the map $A \to E(A)$ is an injection (and therefore an isometry onto its image). Since $E(A)$ is a von Neumann algebra, we have constructed a Banach space W and an isometry $E(A) \simeq W^\vee$ carrying the ultraweak topology on $E(A)$ to the weak $*$-topology on $W^\vee$. The map $\rho : A \to E(A) \simeq W^\vee$ is adjoint to a bounded operator $\rho' : W \to A^\vee$.

Lemma 1. *The map $\rho' : W \to A^\vee$ is an isomorphism of Banach spaces: that is, it admits a continuous inverse.*

By virtue of the open mapping theorem, it will suffice to show that ρ' is an isomorphism of abstract vector spaces. We can regard the elements of W as ultraweakly continuous functionals $E(A) \to \mathbf{C}$. Since A is ultraweakly dense in $E(A)$, such a functional is determined by its restriction to A: this proves that ρ' is injective.

To prove the surjectivity, we must show that every continuous functional $\mu : A \to \mathbf{C}$ extends to an ultraweakly continuous functional on $E(A)$. It clearly suffices to treat the case where μ is Hermitian (that is, where μ satisfies $\mu(a^*) = \overline{\mu(a)}$ for $a \in A$), since $A^\vee$ is the complexification of the the real Banach space consisting of Hermitian functionals. Suppose first that μ is positive: that is, that we have $\mu(a) \geq 0$ for every positive element $a \in A$. In this case, we have seen that $\frac{\mu}{||\mu||}$ is a state of A (so long as $\mu \neq 0$). The result is obvious in this case: every state of A extends to an ultraweakly continuous functional on $E(A)$ by construction. To complete the proof of Lemma 1, it will suffice to prove the following:

Lemma 2. *Let A be a C^*-algebra and let $\mu : A \to \mathbf{C}$ be a continuous functional satisfying $\mu(a^*) = \overline{\mu(a)}$. Then there exist a pair of positive functionals μ_+ and μ_- such that $\mu = \mu_+ - \mu_-$, and*

$$||\mu_+|| + ||\mu_-|| \leq ||\mu||.$$

Example 3. Let A be a commutative C^*-algebra, hence of the form $C^0(X)$ for some compact Hausdorff space X. Then $A^\vee$ can be identified with the Banach space of finite (signed) Baire measures on X. The positive elements of $A^\vee$ are precisely the finite positive measures on X (and the states are precisely the probability measures on the σ-algebra of Baire subsets of X). Lemma 2 expresses the fact that every signed measure can be obtained as a difference of positive measures.

Proof of Lemma 2. We may assume without loss of generality that $||\mu|| \leq 1$. In this case, we will show that $\mu = p\nu_+ + q(-\nu_-)$, where $p + q = 1$ and $\nu_+, \nu_- \in S(A)$ are states of A.

Recall that we can describe $S(A)$ as the set of Hermitian functionals $\nu : A \to \mathbf{C}$ satisfying $||\nu|| \leq 1$ and $\nu(1) = 1$. It follows that $S(A)$ is a closed convex subset of the unit ball of $A^\vee$. Let $S'(A)$ denote the convex hull of the set $S(A) \cup -S(A)$: that is, the image of the map

$$S(A) \times S(A) \times [0, 1] \to A^\vee$$

$$(\nu_+, \nu_-, t) \mapsto t\nu_+ - (1 - t)\nu_-.$$

Since the unit ball of $A^\vee$ is compact in the weak $*$-topology, the set $S(A)$ is weak $*$-compact and therefore $S'(A)$ is also weak $*$-compact, hence a weak $*$-closed subset of $A^\vee$. We wish to show that $\mu \in S'(A)$. Equivalently, we wish to show that μ belongs to the weak closure of $S'(A)$. If not, then there exists a finite sequence of elements of $A_{\mathbb{R}}$ giving a map

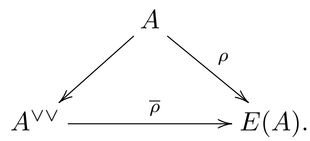
$$q : A^\vee_{\mathbb{R}} \to \mathbb{R}^n$$

such that $q(\mu) \notin q(S'(A))$. Since $q(S'(A))$ is a closed, convex subset of $\mathbb{R}^n$, this means that there exists a hyperplane in $\mathbb{R}^n$ separating $q(S'(A))$ from $q(\mu)$. In this case, we obtain a Hermitian element $a \in A$ and a real number λ such that $\mu(a) > \lambda$, while $\nu(a) \leq \lambda$ for $\nu \in S'(A)$. Since $0 \in S'(A)$, we must have $\lambda > 0$. If $\nu \in A^\vee$ is a state, then we have $\nu(a), -\nu(a) \leq \lambda$ so $|\nu(a)| \leq \lambda$. It follows that $||a|| \leq \lambda$, so that $||\mu(a)|| \leq \lambda$ by virtue of our assumption that $||\mu|| \leq 1$. □

Let us now return to our discussion of the envelope $E(A)$ of a C^*-algebra A. We have isomorphisms $E(A) \simeq W^\vee$ and $W \simeq A^\vee$, which together give an isomorphism of Banach spaces

$$\overline{\rho} : A^{\vee\vee} \simeq W^\vee \simeq E(A).$$

By construction, this isomorphism fits into a commutative diagram



Moreover, it carries the ultraweak topology on $E(A)$ to the weak $*$-topology on $A^{\vee\vee} \simeq W^{\vee}$.

Remark 4. With more effort, one can show that the isomorphism of Banach spaces $\overline{\rho}$ is actually an isometry (by construction, $\overline{\rho}$ has operator norm ≤ 1, and Lemma 2 shows that it is an isometry when restricted to the real subspace consisting of Hermitian elements).

To describe the primary example of the above paradigm, we need a few remarks about nonunital algebras. Suppose that A is a nonunital $*$-algebra and that V is a Hilbert space representation of A (as a nonunital algebra; if A happens to have a unit $1 \in A$, we do not require that it acts by the identity on V). Let V_0 denote the subspace of V consisting of those vectors which are annihilated by each element of a. Note that $v \in V_0$ if and only if $(a^* v, w) = 0$ for all $a \in A$ and $w \in W$. This is true if and only if $(v, aw) = 0$: that is, if and only if V_0 is orthogonal to the subspace $AV \subseteq V$.

Definition 5. Let A be a nonunital $*$-algebra. A representation V of A is said to be *nondegenerate* if either of the following equivalent conditions holds:

(a) The subspace V_0 is trivial: that is, no nonzero element of V is annihilated by the entire algebra A.

(b) The subspace AV is dense in V.

We have the following slightly stronger form of von Neumann's theorem:

Theorem 6. *Let $A \subseteq B(V)$ be a nonunital $*$-subalgebra, and assume V is a nondegenerate representation of A. Then A is ultra-strongly dense in its double commutant A''. In particular, the identity* $\mathrm{id} : V \to V$ *is an ultra-strong limit of elements of A.*

Proof. As in the proof of the unital version, we can replace V by $V^{\oplus\infty}$ and thereby reduce to showing that for every $f \in A''$ and every vector $v \in V$, the vector $f(v)$ belongs to the closure of Av. Again, we let e denote orthogonal projection onto $\overline{Av}$ and observe that $e \in A'$, so that $ef(v) = fe(v)$. To prove that $f(v) \in \overline{Av}$, it suffices to show $v \in \overline{Av}$. Let W be the subspace of V generated by $\overline{Av}$ and v. Since V is a nondegenerate representation of A, so is W. It follows that $v \in W = \overline{AW} \subseteq \overline{A(\mathbf{C}v + Av)} \subseteq \overline{Av}$, as desired. $\square$

Example 7. Let $A \subseteq B(V)$ be a von Neumann algebra and let $I \subseteq A$ be a left ideal. We say that I is a $*$-*ideal* if it is closed under the operation $a \mapsto a^*$ (in which case it follows that I is also a right ideal, hence a two-sided ideal of A). Suppose further that I is closed in the ultraweak topology. Then $V \simeq V_0 \oplus \overline{IV}$, where V_0 is the subspace of V consisting of elements which are annihilated by each element of I. Note that $\overline{IV}$ is a nondegenerate representation of the nonunital $*$-algebra I. Since I is ultraweakly (and therefore ultrastrongly) closed in $B(\overline{IV})$, we conclude that I is a von Neumann algebra in $B(\overline{IV})$: in particular, it contains the identity element of $B(\overline{IV})$ (which we can identify with the element $e \in B(V)$ given by orthogonal projection onto $\overline{IV}$.