Math 261y: von Neumann Algebras (Lecture 7)

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Let V be a Hilbert space. In the last lecture, we defined the subspace $B^{\mathrm{tc}}(V) \subseteq B(V)$ of trace-class operators: an operator is trace-class if it has the form

$$u \mapsto \sum (u, w_i) v_i$$

where $\sum ||v_i||^2 < \infty$ and $\sum ||w_i||^2 < \infty$. To such an operator we can associate a well-defined trace $\sum (v_i, w_i)$, and the trace pairing $(f, g) \mapsto \mathrm{tr}(fg)$ determines an isometric isomorphism of $B(V)$ with the dual $B^{\mathrm{tc}}(V)^{\vee}$.

For any Banach space W, the dual space $W^{\vee}$ inherits a topology, called the *weak $*$-topology*: it is the coarsest topology for which the linear functionals $\mu \mapsto \mu(w)$ are continuous, for each $w \in W$. Taking $W = B^{\mathrm{tr}}(V)$ (so that $W^{\vee} \simeq B(V)$), we obtain a topology on the operator algebra $B(V)$. Using Remark **??**, we see that this is the coarsest topology for which the functionals

$$f \mapsto \sum_i (f(v_i), w_i)$$

are continuous, for all sequences of vectors v_i and w_i satisfying

$$\sum ||v_i||^2 < \infty \qquad \sum ||w_i||^2 < \infty.$$

It follows that the weak $*$-topology on $B(V)$ coincides with the ultraweak topology introduced in Lecture 5.

Corollary 1. *Let $A \subseteq B(V)$ be a von Neumann algebra. Then there exists an isometry $A \simeq W^{\vee}$, for some Banach space W.*

Since A is closed in the ultraweak topology on $B(V)$ (which coincides with the weak $*$-topology), this follows from the following more general claim:

Proposition 2. *Let M be a Banach space and let $A \subseteq M^{\vee}$ be a subspace which is closed in the weak $*$-topology. Then $A \simeq W^{\vee}$ for some Banach space W.*

Proof. Take K to be the subspace of M given by the kernel of those functionals belonging to A, and let $W = M/K$. We have an exact sequence of Banach spaces

$$0 \to K \to M \to W \to 0$$

which gives an exact sequence of dual spaces

$$0 \to W^{\vee} \to M^{\vee} \to K^{\vee} \to 0.$$

It will therefore suffice to show that $A = \ker(M^{\vee} \to K^{\vee})$: that is, that a linear functional $f \in M^{\vee}$ belongs to A if and only if it vanishes on K. Let $\mu : M \to \mathbf{C}$ be a continuous functional which vanishes on K; we wish to show that $\mu \in A$. For this, it suffices to show that μ belongs to the weak $*$-closure of A. That is, we must show that for every sequence of elements $x_1, \ldots, x_n \in M$ and every $\epsilon > 0$, there exists $\mu' \in A$ such that

$$|\mu(x_i) - \mu'(x_i)| < \epsilon$$

for each x_i. In fact, we claim that we can choose μ' such that $\mu(x_i) = \mu'(x_i)$. For this, let us consider the map

$$T : M^\vee \to \mathbb{R}^n$$

given by $\mu' \mapsto (\mu'(x_1), \mu'(x_2), \ldots, \mu'(x_n))$. If $T(\mu) \notin T(A)$, then there exists a linear functional on $\mathbb{R}^n$ which vanishes on $T(A)$ but not on $T(\mu)$. This linear functional is given by some linear combination $\vec{x} = \sum \lambda_i x_i$. Then $\vec{x} \in K$ but $\mu(x) \neq 0$, contradicting our assumption on μ. □

We will later show that the Banach space W appearing in Corollary 1 is unique. The isomorphism $A \simeq W^\vee$ allows us to regard A as equipped with the weak $*$-topology, which will coincide with the restriction of the ultraweak topology on $B(V)$.

Definition 3. Let $A \subseteq B(V)$ and $A' \subseteq B(V')$ be von Neumann algebras. A *morphism of von Neumann algebras* from A to A' is a $*$-algebra homomorphism $\phi : A \to A'$ which is continuous for the ultraweak topologies.

A *von Neumann algebra representation* of $A \subseteq B(V)$ is a von Neumann algebra homomorphism $\rho : A \to B(W)$, for some Hilbert space W. That is, a von Neumann algebra representation is an action of A on a Hilbert space W satisfying the condition that

$$(av, w) = (v, a^* w)$$

and that the map

$$a \mapsto \sum (av_i, w_i)$$

is continuous for the ultraweak topology on A, whenever $v_i, w_i \in W$ are vectors satisfying $\sum ||v_i||^2 < \infty$ and $\sum ||w_i||^2 < \infty$. We will refer to this last condition as *ultraweak continuity*.

Our goal in this lecture is to prove the following:

Theorem 4. *Let $A \subseteq B(V)$ be a von Neumann algebra, and suppose we are given a representation W of the underlying C^*-algebra of A. Then W is a von Neumann algebra representation of A if and only if it can be obtained as a direct summand of a (possibly infinite) direct sum of copies of V.*

In other words, a von Neumann algebra $A \subseteq B(V)$ has essentially only one representation V: all other representations can be obtained from V by means of obvious operations.

Definition 5. Let A be a C^*-algebra. A representation V of A is said to be *cyclic* if there exists a vector $v \in V$ such that Av is dense in V.

Proposition 6. *Let A be an arbitrary C^*-algebra. Then every representation V of A can be obtained as an orthogonal direct sum of cyclic representations.*

Proof. Let S denote the collection of all closed subspaces of V which are A-invariant and cyclic. Let T be the collection of all subsets $S_0 = \{V_\alpha\}$ of S which are mutually orthogonal: that is, V_α is orthogonal to V_β for every pair of distinct elements $V_\alpha, V_\beta \in S_0$. We regard T as a partially ordered set with respect to inclusions. Using Zorn's lemma, we see that T contains a maximal element S_0. Let W denote the closed subspace of V generated by the subspaces $V_\alpha \in S_0$. If $W = V$, we are done. Otherwise, we can choose a vector $v \in V$ belonging to the orthogonal complement of W. Since V is a $*$-representation of A, we see that the entire orbit Av is orthogonal to W. It follows that $S_0 \cup \{\overline{Av}\}$ belongs to T, contradicting the maximality of S_0. □

Using Proposition 6, we can reduce Theorem 4 to the following assertion:

Proposition 7. *Let $A \subseteq B(V)$ be a von Neumann algebra, and let W be a cyclic von Neumann algebra representation of A. Then W is isomorphic to a direct summand of the countable orthogonal direct sum $V^{\oplus\infty} = V \oplus V \oplus \cdots$.*

Fix a cyclic vector $w \in W$, so that Aw is dense in W. Since W is a von Neumann algebra representation, the functional $\mu : A \to \mathbf{C}$ given by $\mu(a) = (aw, w)_W$ is continuous for the ultraweak topology. We may therefore write

$$\mu(a) = \sum (av_i, v'_i)_V$$

for some elements $v_i, v'_i \in V$ satisfying

$$\sum ||v_i||^2 < \infty \qquad \sum ||v'_i||^2 < \infty.$$

Let us regard the sequences $\{v_i\}$ and $\{v'_i\}$ as elements of the direct sum $V^{\oplus\infty}$. Replacing A by its image in $B(V^{\oplus\infty})$, we are reduced to proving the following:

Proposition 8. *Let $A \subseteq B(V)$ be a von Neumann algebra. Let W be a representation of A with a cyclic vector w, let $\mu : A \to \mathbf{C}$ be given by $\mu(a) = (aw, w)_W$, and suppose there exist vectors $v, v' \in V$ with $\mu(a) = (av, v')_V$. Then W is isomorphic (as a representation of A) to a direct summand of the Hilbert space V.*

Let us first indulge in a slight digression. Let A be any C^*-algebra acting on a Hilbert space W, and let $w \in W$ be a unit vector. We have seen that the map $\mu : A \to \mathbf{C}$ given by $\mu(a) = (aw, w)$ is a state on A. Given this state, we can construct a representation V_μ by completing A with respect to the inner product $\langle a, b \rangle = \mu(b^* a)$. The construction $a \mapsto aw$ then extends to an isometric embedding $V_\mu \to W$, and therefore gives a direct sum decomposition $W \simeq V_\mu \oplus V_\mu^\perp$. If $w \in W$ is a cyclic vector, we get an isomorphism $W \simeq V_\mu$.

To prove Proposition 8, we may assume without loss of generality that $w \in W$ is a unit vector, so that μ is a state $W \simeq V_\mu$ by the analysis given above. To realize W as a direct summand of V, it will suffice to find a vector $u \in V$ such that $\mu(a) = (au, u)_V$.

Note that μ is a *positive* linear functional: that is, we have $\mu(a) \geq 0$ whenever $a \in A$ is a positive element. To prove this, we can write $a = b^* b$, so that $\mu(a) = (b^* b w, w)_W = (bw, bw)_W \geq 0$. For each positive element $a \in A$, we have

$$\begin{aligned} \mu(a) &\leq \mu(a) + \frac{1}{4}(a(v - v'), v - v')_V \\ &= \mu(a) + \frac{1}{4}(av, v)_V + \frac{1}{4}(av', v')_V - \frac{1}{4}(av, v')_V - \frac{1}{4}(av', v)_V. \end{aligned}$$

Since a is Hermitian, we have $(av', v) = (v', av) = \overline{(av, v')} = \overline{\mu(a)} = \mu(a)$. We therefore obtain

$$\begin{aligned} \mu(a) &\leq \mu(a) + \frac{1}{4}(av, v)_V + \frac{1}{4}(av', v')_V - \frac{1}{4}\mu(a) - \frac{1}{4}\mu(a) \\ &= \frac{1}{4}(av, v)_V + \frac{1}{4}(av', v')_V + \frac{1}{4}(av, v') + \frac{1}{4}(av', v) \\ &= \frac{1}{4}(a(v + v'), v + v'). \end{aligned}$$

Since μ is positive, the construction $\mu(b^* a)$ determines an inner product on A. The Cauchy-Schwartz inequality then gives

$$\mu(b^* a)^2 \leq \mu(b^* b)\mu(a^* a) \leq \frac{1}{16}(b^* b(v+v'), v+v')_V (a^* a(v+v'), v+v')_V = \frac{1}{16}(b(v+v'), b(v+v'))_V (a(v+v'), a(v+v'))_V.$$

Equivalently, we have

$$\mu(b^* a) \leq \frac{1}{4}||b(v + v')||\,||a(v + v')||.$$

Let $V_0 \subseteq V$ be the closure of $A(v+v')$. It follows from the above analysis that the formula

$$\langle a(v+v'), b(v+v')\rangle = \mu(b^* a)$$

extends continuously to an inner product $\langle , \rangle : V_0 \times V_0 \to \mathbf{C}$ satisfying $\langle x, y\rangle \leq \frac{1}{4}||x||\,||y||$. For fixed x, the map $y \mapsto \langle x, y\rangle$ is a continuous antilinear functional of norm $\leq \frac{1}{4}||x||$, so that $\langle x, y\rangle = (f(x), y)$ for some $f(x) \in V_0$ with $||f(x)|| \leq \frac{1}{4}||x||$. The map $x \mapsto f(x)$ is evidently linear, and has norm $\leq \frac{1}{4}$. We conclude that

$$\mu(b^* a) = \langle a(v+v'), b(v+v')\rangle = (fa(v+v'), b(v+v'))_V.$$

In particular, we get

$$(b^* fa(v+v'), c(v+v'))_V = (fa(v+v'), bc(v+v')) = \mu(c^* b^* a) = (fb^* a(v+v'), c(v+v'))).$$

By continuity, we deduce that fb^* and $b^* f$ agree on the whole of V_0, so that $f \in B(V_0)$ commutes with the action of A. The operator f is evidently positive (since $\langle , \rangle$ is positive semidefinite), so it admits a unique positive square root $f^{1/2}$ which also commutes with the action of A. We then have

$$\mu(b^* a) = (fa(v+v'), b(v+v'))_V = (af^{1/2}(v+v'), bf^{1/2}(v+v')),$$

so that W is isomorphic to the cyclic subrepresentation of V generated by the vector $f^{1/2}(v+v')$.