Math 261y: von Neumann Algebras (Lecture 6)

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Let us begin this lecture with a short review of the theory of trace-class operators. Let V be a Hilbert space. For every pair of nonzero elements $v, w \in V$, we introduce a formal symbol $e_{v,w}$. Let X denote the vector space consisting of formal sums

$$\sum_{v,w \in V - \{0\}} \lambda_{v,w} e_{v,w}$$

such that $\sum |\lambda_{v,w}| \, ||v|| \, ||w|| < \infty$ (this implies, in particular, that $\lambda_{v,w} = 0$ for all but countably many pairs (v, w)). We can regard X as a Banach space, with norm given by

$$|| \sum \lambda_{v,w} e_{v,w} || = \sum |\lambda_{v,w}| \, ||v|| \, ||w||.$$

By convention, we let $e_{v,w} = 0 \in X$ if either v or w is the zero element of V.

For $v, w \in V$, we define an element $\rho(e_{v,w}) \in B(V)$ by the formula

$$\rho(e_{v,w})(u) = (u, w)v.$$

It is easy to see that the norm of the operator $\rho(e_{v,w})$ is exactly $||v|| \, ||w||$. Consequently, we can extend ρ to a linear map $\rho : X \to B(V)$ having norm ≤ 1; formally we have

$$\rho(\sum_{v,w} \lambda_{v,w} e_{v,w}) = \sum_{v,w} \lambda_{v,w} \rho(e_{v,w})$$

(where this sum is convergent with respect to the norm topology on $B(V)$.

Definition 1. We say that an operator $f \in B(V)$ is *trace class* if it belongs to the image of the linear map $\rho : X \to B(V)$. We let $B^{\mathrm{tc}}(V)$ denote the linear subspace of $B(V)$ consisting of trace class operators.

Proposition 2. *Let $f \in B(V)$ be a trace-class operator. Then f^* is also trace-class. For each $g \in B(V)$, we the operators gf and fg are also trace class. That is, $B^{\mathrm{tc}}(V)$ is a two-sided $*$-ideal in $B(V)$.*

Proof. Write $f = \rho(\sum \lambda_{v,w} e_{v,w})$. Then

$$f^* = \rho(\sum \overline{\lambda}_{w,v} e_{w,v}) \qquad gf = \rho(\sum \lambda_{v,w} e_{g(v),w}) \qquad fg = \rho(\sum \lambda_{v,w} e_{v,g^*(w)}).$$

□

Warning 3. The subspace $B^{\mathrm{tc}}(V)$ is not closed with respect to any of the natural topologies on $B(V)$. The collection of finite rank operators (generated as a vector space by $\rho(e_{v,w})$ for $v, w \in V$) is norm-dense in $B^{\mathrm{tc}}(V)$. The closure of $B^{\mathrm{tc}}(V)$ with respect to the norm topology is the space of compact operators on V. With respect to any of the other topologies described in the previous lecture, the closure of $B^{\mathrm{tc}}(V)$ is all of $B(V)$. To prove this, it suffices to show that $B^{\mathrm{tc}}(V)$ is closed in $B(V)$ with respect to the ultrastrong topology. Let $f \in B(V)$ by any operator, and choose an orthonormal basis $\{v_\alpha\}_{\alpha \in A}$ for V. For each finite subset $A_0 \subseteq A$, let $f_{A_0} = \rho(\sum_{\alpha \in A_0} e_{v_\alpha, f(v_\alpha)})$. We claim that the net $\{f_{A_0}\}_{A_0 \subseteq A}$ converges ultrastrongly to

f. To prove this, it suffices to show that for every sequence $w_i \in V$ with $\sum ||w_i||^2 < \infty$, the real numbers $\sum ||f(w_i) - f_{A_0}(w_i)||^2$ converge to zero as A_0 becomes large. We can rewrite this sum as $\sum ||f(\pi w_i)||^2$, where π denotes the projection onto the orthogonal complement of the span of $\{v_\alpha\}_{\alpha \in A_0}$. Since f is bounded, we are reduced to showing that the sum $\sum ||\pi w_i||^2$ converges to zero as A_0 becomes large, which we leave to the reader.

Despite Warning 3, we can view $B^{\mathrm{tc}}(V)$ as a Banach space, via the isomorphism $B^{\mathrm{tc}}(V) \simeq X/\ker(\rho)$. This isomorphism determines a norm on $B^{\mathrm{tc}}(V)$, called the *trace class norm*, which we will denote by $f \mapsto ||f||_1$. By definition, we have

$$||f||_1 = \inf\{\sum |\lambda_{v,w}| \, ||v|| \, ||w|| : f = \rho(\sum \lambda_{v,w} e_{v,w}\}.$$

With more effort, one can show that this infimum is actually achieved, but we will not need to know this.

Let $f \in B(V)$ be a bounded operator. We define a linear functional $\chi_f : X \to \mathbf{C}$ by the formula

$$\chi_f(\sum \lambda_{v,w} e_{v,w}) = \sum \lambda_{v,w}(f(v), w).$$

Note that this sum converges and is bounded in norm by

$$||f|| \sum |\lambda_{v,w}| \, ||v|| \, ||w||.$$

It follows that χ_f is a continuous functional of norm $\leq ||f||$. In fact, we claim that the norm of χ_f is exactly the norm $||f||$. To prove this, suppose that $||f|| > \epsilon$, so that there exists a unit vector $v \in V$ such that $||f(v)|| > \epsilon$. Then $e_{v,f(v)} \in X$ has norm equal to $||f(v)||$, so that

$$||\chi_f|| \geq \frac{|\chi_f(e_{v,f(v)})|}{||f(v)||} = \frac{|(f(v), f(v))|}{||f(v)||} = ||f(v)|| > \epsilon.$$

The upshot of this discussion is that the construction $f \mapsto \chi_f$ determines an isometric embedding of Banach spaces $B(V) \to X^\vee$. Here $X^\vee$ denote the dual of X.

Lemma 4. *Let $f \in B(V)$. Then the linear functional $\chi_f : X \to \mathbf{C}$ annihilates* $\ker(\rho)$.

Proof. Fix an element $\sum \lambda_{v,w} e_{v,w}$ belonging to $\ker(\rho) \subseteq X$. We wish to show that for each $f \in B(V)$, the sum $\sum \lambda_{v,w}(f(v), w)$ vanishes. Write $\lambda_{v,w} = \mu_{v,w} |\lambda_{v,w}|$ where $\mu_{v,w}$ has absolute value 1. Then

$$\sum \lambda_{v,w}(f(v), w) = \sum \mu_{v,w}(f((\frac{|\lambda_{v,w}| \, ||w||}{||v||})^{1/2} v), (\frac{|\lambda_{v,w}| \, ||v||}{||w||})^{1/2} w).$$

Since we have

$$\sum ||(\frac{|\lambda_{v,w}| \, ||w||}{||v||})^{1/2} v||^2 = \sum |\lambda_{v,w}| \, ||v|| \, ||w|| < \infty$$

$$\sum ||(\frac{|\lambda_{v,w}| \, ||v||}{||w||})^{1/2} w||^2 = \sum |\lambda_{v,w}| \, ||v|| \, ||w|| < \infty,$$

we conclude that the expression $\sum \lambda_{v,w}(f(v), w)$ is an ultraweakly continuous function of f. Since we have seen that the collection of finite rank operators is dense with respect to the ultraweak topology (or even with respect to the ultrastrong topology), it will suffice to check the equality when f is an operator of finite rank. We may therefore assume that $f = \rho(e_{v',w'})$ for some vectors $v', w' \in V$. Then $\chi_f : X \to \mathbf{C}$ is given by

$$\sum \lambda_{v,w} e_{v,w} \mapsto \sum \lambda_{v,w}(v, w')(v', w) = (\rho(\sum_{v,w} \lambda_{v,w} e_{v,w})(v'), w'),$$

which vanishes by assumption. □

The exact sequence of Banach spaces

$$0 \to \ker(\rho) \to X \to B^{\text{tc}}(V) \to 0$$

determines an exact sequence of dual spaces

$$0 \to B^{\text{tc}}(V)^{\vee} \to X^{\vee} \to \ker(\rho)^{\vee} \to 0.$$

It follows from Lemma 4 that the construction $f \mapsto \chi_f$ factors through $B^{\text{tc}}(V)^{\vee}$.

Proposition 5. *The construction $f \mapsto \chi_f$ induces an isometric isomorphism $B(V) \to B^{\text{tc}}(V)^{\vee}$.*

Proof. Let $\mu : X \to \mathbf{C}$ be any continuous functional which annihilates $\ker(\rho)$; we wish to show $\mu = \chi_f$ for some $f \in B(V)$. We may assume without loss of generality that μ has norm ≤ 1, so that $|\mu(e_{v,w})| \leq ||v|| \, ||w||$ for $v, w \in V$. Fix an element $v \in V$. Since $e_{v,w+w'} - e_{v,w} - e_{v,w'}$ and $e_{v,\lambda w} - \overline{\lambda} e_{v,w}$ belong to $\ker(\rho)$, we see that the construction

$$w \mapsto \mu(e_{v,w})$$

is a $\mathbf{C}$-antilinear functional on V, having norm $\leq ||v||$. It is therefore given by $w \mapsto (f(v), w)$ for some $f(v) \in V$ with $||f(v)|| \leq ||v||$. Since $e_{v+v',w} - e_{v,w} - e_{v',w}$ and $e_{\lambda v,w} - \lambda e_{v,w}$ belong to the kernel of ρ, we deduce that f is linear, so that $f \in B(V)$. By construction, we have

$$\mu(e_{v,w}) = \chi_f(e_{v,w})$$

for $v, w \in V$. Since the linear span of the elements $e_{v,w}$ is dense in X, we conclude that $\mu = \chi_f$. □

The isomorphism $B(V) \to B^{\text{tc}}(V)^{\vee}$ carries the unit element $\text{id} \in B(V)$ to a continuous functional $\chi_{\text{id}} : B^{\text{tc}}(V) \to \mathbf{C}$, given by

$$\rho(\sum \lambda_{v,w} e_{v,w}) \mapsto \sum \lambda_{v,w} (v, w).$$

We will denote this map by $\text{tr} : B^{\text{tc}}(V) \to \mathbf{C}$, and call it the *trace map* on $B^{\text{tc}}(V)$. For $f \in B(V)$, we see that χ_f is given by

$$\rho(\sum \lambda_{v,w} e_{v,w}) \mapsto \sum \lambda_{v,w} (f(v), w) = \text{tr}\, \rho(\sum \lambda_{v,w} e_{f(v),w}) = \text{tr}(f \rho(\sum \lambda_{v,w} e_{v,w})).$$

In other words, we have $\chi_f(g) = \text{tr}(fg)$ for $f \in B(V)$, $g \in B^{\text{tr}}(V)$. A similar calculation gives $\chi_f(g) = \text{tr}(gf)$, so that $\text{tr}(gf) = \text{tr}(fg)$.

Remark 6. For every element $\sum_{v,w} \lambda_{v,w} e_{v,w}$ belonging to X, we have

$$\rho(\sum_{v,w} \lambda_{v,w} e_{v,w}) = \rho(\sum_{v,w} e_{\mu_{v,w} (\frac{|\lambda_{v,w}| \, ||w||}{||v||})^{1/2} v, (\frac{|\lambda_{v,w}| \, ||v||}{||w||})^{1/2} w},$$

where $\mu_{v,w}$ is as in the proof of Lemma 4. It follows that every element of $B^{\text{tr}}(V)$ can be written as

$$\rho(\sum_i e_{v_i, w_i}),$$

for some sequences of vectors v_i and w_i satisfying

$$\sum ||v_i||^2 < \infty \qquad \sum ||w_i||^2 < \infty.$$

Conversely, every such sum determines an element of $B^{\text{tr}}(V)$ (by the Cauchy-Schwartz inequality).