Math 261y: von Neumann Algebras (Lecture 5)

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In this lecture, we will state and prove von Neumann's double commutant theorem. First, let's establish a bit of notation. Let V be a Hilbert space, and let $B(V)$ denote the Hilbert space of bounded operators on V. We will consider several topologies on $B(V)$:

(1) The *norm topology*, which has a subbasis (in fact, a basis) of open sets given by $\{x \in B(V) : ||x - x_0|| < \epsilon\}$ where $x_0 \in B(V)$ and ϵ is a positive real number.

(2) The *strong topology*, which has a subbasis of open sets given by $\{x \in B(V) : ||x(v) - x_0(v)|| < \epsilon\}$, where $x_0 \in B(V)$, $v \in V$, and ϵ is a positive real number.

(3) The *weak topology*, which has a subbases of open sets given by $\{x \in B(V) : |(x(v) - x_0(v), w)| < \epsilon\}$ where $x_0 \in B(V)$, $v, w \in V$, and ϵ is a positive real number.

Each of these topologies is more coarse than the previous. That is, we have norm convergence $\Rightarrow$ strong convergence $\Rightarrow$ weak convergence.

Warning 1. With respect to the norm topology, $B(V)$ is a metric space, so its topology is determined by the collection of convergent sequences. However, the strong and weak topologies do not have this property. However, the unit ball of $B(V)$ is metrizable in either topology if V is a separable Hilbert space.

Warning 2. The multiplication map

$$B(V) \times B(V) \to B(V)$$
$$(x, y) \mapsto xy$$

is not continuous with respect to either the strong or weak topology. However, it is separately continuous in each variable. Let us prove this for the weak topology. Fix $x, y_0 \in B(v)$ and consider a subbasic open neighborhood of $\{y \in B(v) : |(y(v) - y_0(v), w)| < \epsilon\}$ in the weak topology. Taking the inverse image under left multiplication by x, we get the subbasic open set

$$\{y \in B(V) : |(xy(v) - xy_0(v), w)| < \epsilon\} = \{y \in B(V) : |(y(v) - y_0(v), x^*(w)| < \epsilon\}.$$

Taking the inverse image under right multiplication by x, we get the subbasic open set $\{y \in B(V) : |(y(x(v)) - y_0(x(v)), w| < \epsilon\}$.

Warning 3. The map $x \mapsto x^*$ is continuous with respect to the weak topology, since the condition $|(x^*(v) - x_0^*(v), w)| < \epsilon$ can be rewritten as $|(x(w) - x_0(w), v)| < \epsilon$. However, it is not continuous with respect to the strong topology.

We are now ready to state the theorem:

Theorem 4 (von Neumann)**.** *Let V be a Hilbert space, let $A \subseteq B(V)$ be a $*$-subalgebra. The following conditions are equivalent:*

(1) *The algebra A is its own double commutant A''.*

(2) *The algebra A is closed with respect to the weak topology on $B(V)$.*

(3) *The algebra A is closed with respect to the strong topology on $B(V)$.*

We say that a $*$-subalgebra $A \subseteq B(V)$ is a *von Neumann algebra* if A satisfies the equivalent conditions of Theorem 4.

The implication (2) $\Rightarrow$ (3) is tautologous. The implication (1) $\Rightarrow$ (2) is not difficult: for any subset $S \subseteq B(V)$, we have

$$S' = \bigcap_{x \in S} \{y \in B(V) : xy - yx = 0.\}$$

This is closed in the weak topology, since the function $x \mapsto xy - yx$ is continuous with respect to the weak topology on $B(V)$. The real content is the implication (3) $\Rightarrow$ (1). We can reformulate this condition as follows:

Proposition 5. *Let $A \subseteq B(V)$ be a $*$-subalgebra. Then A is dense in A'' with respect to the strong topology.*

Let us first prove a simple version of Proposition 5. Suppose that we are given an element $x_0 \in A'' \subseteq B(V)$, and consider a subbasic open neighborhood of x_0

$$U = \{x \in B(V) : ||x(v) - x_0(v)|| < \epsilon\}.$$

We wish to prove that U contains an element of A. In other words, we wish to show that $x_0(v)$ belongs to the closure of the subset $Av \subseteq V$. Let us denote this closure by W, and let $e : V \to V$ be the orthogonal projection onto W. Since W is A-invariant and A is an $*$-algebra, we conclude that $W^{\perp}$ is also A-invariant and therefore the projection e belongs to the commutant A'. Then e commutes with $x_0 \in A''$, so that $ex_0(v) = x_0 e(v) = x_0(v)$. This proves that $x_0(v)$ belongs to $W = \overline{Av}$, as desired.

Of course, this is not sufficient to prove Proposition 5, because sets of the form $\{x \in B(V) : ||x(v) - x_0(v)|| < \epsilon\}$ only form a subbasis for the strong topology on $B(V)$, not a basis. To prove density, we need to show that for any finite sequence of vectors $v_1, \ldots, v_n$ in V (and any real number $\epsilon > 0$, we can find an element $x \in A$ which simultaneously satisfies the inequalities

$$||x(v_i) - x_0(v_i)|| < \epsilon.$$

for $1 \leq i \leq n$. For this, we use a trick: let $V^{\oplus n}$ denote a direct sum of n copies of the Hilbert space V. The algebra A acts on $V^{\oplus n}$ via the formula

$$x(w_1, \ldots, w_n) = (xw_1, \ldots, xw_n).$$

This construction determines an embedding of A into $B(V^{\oplus n})$; let us denote the image of this embedding by $A(n)$. Note that $B(V^{\oplus n})$ can be identified with the algebra $M_n(B(V))$ of n-by-n matrices with values in $B(V)$. Under this identification, the commutant $A(n)'$ corresponds to those n-by-n matrices whose entries belong to the commutant A'. The double commutant $A(n)''$ can the be identified with image of A'' in $B(V^{\oplus n})$: that is, we have $A(n)'' = A''(n)$. The argument given above shows that for every vector $(v_1, \ldots, v_n) \in V^{\oplus n}$, the element $(x_0 v_1, \ldots, x_0 v_n)$ belongs to the closure of the subspace $A(n)(v_1, \ldots, v_n) \subseteq V^{\oplus n}$. In other words, for each $\epsilon > 0$ we can find $x \in A$ such that

$$\sum_{1 \leq i \leq n} ||x(v_i) - x_0(v_i)||^2 < \epsilon^2,$$

so that in particular $||x(v_i) - x_0(v_i)|| < \epsilon$ for each i. This completes the proof of Theorem 4.

We now note that the above argument actually gives something a little stronger. There is no need to take a sum of *finitely many* copies of V: we could just as well replace V by an infinite direct sum $V \oplus V \oplus V \oplus \cdots$. We therefore obtain the following:

Proposition 6. *Let $A \subseteq B(V)$ be a $*$-subalgebra. For any element $x_0 \in A''$, any real number $\epsilon > 0$, and any sequence of vectors $v_1, v_2, \ldots \in V$ with $\sum_i ||v_i||^2 < \infty$, there exists $x \in A$ such that*

$$\sum_i ||x(v_i) - x_0(v_i)||^2 < \epsilon.$$

Motivated by Proposition 5, we introduce the following definition:

Definition 7. Let V be a Hilbert space. The *ultrastrong topology* on $B(V)$ has a basis consisting of sets of the form

$$\{x \in B(V) : \sum_i ||x(v_i) - x_0(v_i)||^2 < \epsilon\},$$

where $x_0 \in B(V)$, $\epsilon > 0$, and $\{v_i\}$ is a sequence of elements of v satisfying $\sum_i ||v_i||^2 < \infty$.

We have proven:

Proposition 8. *Any von Neumann algebra $A \subseteq B(V)$ is closed in the ultrastrong topology.*

We can also endow $B(V)$ with the *ultraweak* topology, which has a subbasis given by sets of the form

$$\{x \in B(V) : |\sum_i (x(v_i) - x_0(v_i), w_i)| < \epsilon\}$$

where $\sum ||v_i||^2 < \infty$ and $\sum ||w_i||^2 < \infty$. Every subset of $B(V)$ which is closed for the ultrastrong topology is also closed for the ultraweak topology, so that von Neumann algebras are ultraweakly closed. Conversely, any ultraweakly closed subset of $B(V)$ is weakly closed, so that an ultraweakly closed $*$-subalgebra of $B(V)$ is a von Neumann algebra.

Remark 9. We can describe the ultraweak and ultrastrong topology on $B(V)$ as the restrictions of the weak and strong topologies on $B(V^{\oplus\infty})$ under the embedding

$$B(V) \to B(V^{\oplus\infty}).$$

Remark 10. One argument in favor of the ultrastrong and ultraweak topologies is that they are *intrinsic* to the von Neumann algebra A being studied: that is, they do not depend on a chosen embedding $A \hookrightarrow B(V)$. The strong and weak topologies do not have this property.

Remark 11. The analogue of Theorem 4 for the norm topology is false. For example, let X be a compact Hausdorff space equipped with a regular Borel measure. Then the C^*-algebra $C^0(X)$ of continuous functions on X acts on the Hilbert space $V = L^2(X)$. One can show that the commutant of $C^0(X)$ is the space $L^\infty(X)$ of essentially bounded functions on X, which is generally larger than $C^0(X)$, despite the fact that $C^0(X)$ is closed in the norm topology of $B(V)$.