# Math 261y: von Neumann Algebras (Lecture 3)

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In the last lecture, we introduced the notion of a *positive* element of a $C^*$-algebra $A$. Our first goal in this lecture is to prove the following:

**Proposition 1.** *Let $A$ be a $C^*$-algebra. Then an element $x \in A$ is positive if and only if we can write $x = y^*y$ for some $y \in A$.*

*Proof.* The "if" direction is obvious (in fact, we can take $y = y^*$ to be another positive element of $A$, as we saw in the last lecture). So let us suppose that $x = y^*y$; we wish to show that $x$ is positive. Write $x = x_+ + x_-$ as in the previous lecture; we wish to prove that $x_- = 0$. Let $y' = yx_-$, so that

$$y'^*y' = x_-^*y^*yx_- = x_-(x_+ + x_-)x_- = x_-^3.$$

Note that $x_-^3$ is negative, and vanishes if and only if $x_-$ vanishes (we can check this in the commutative $C^*$-algebra generated by $x_-$). Replacing $y$ by $y'$, we are reduced to proving that if the product $y^*y$ is negative, then $y^*y = 0$.

We next observe that our claim does not depend on whether we consider the product $y^*y$ or $yy^*$. This is a consequence of the following more general observation:

**Lemma 2.** *Let $A$ be a Banach algebra (or any algebra) containing elements $a$ and $b$. Then $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$*

*Proof.* We must show that if $\lambda$ is a nonzero complex number, then $ab - \lambda$ is invertible if and only if $ba - \lambda$ is invertible. We will prove that "if" direction; the converse follows by symmetry. Dividing by $\lambda$, we can reduce to the case $\lambda = 1$. Let $ba - 1$ be invertible, so that there is an element $u \in A$ with

$$bau = u + 1 = uba.$$

Let $v = aub - 1$. Then

$$abv = ab(aub) - ab = a(bau)b - ab = a(u + 1)b - ab = aub = v + 1$$

$$vab = (aub)ab - ab = a(uba)b - ab = a(u + 1)b - ab = aub = v + 1$$

so that $v$ is an inverse of $ab - 1$. □

It follows that if $y^*y$ is negative, $yy^*$ is also negative. Then $y^*y + yy^*$ is also negative. However, writing $y = \Re(y) + \Im(y)$, we obtain

$$\begin{aligned} y^*y + yy^* &= (\Re(y) - i\Im(y))(\Re(y) + i\Im(y)) + (\Re(y) + i\Im(y))(\Re(y) - i\Im(y)) \\ &= 2\Re(y)^2 + 2\Im(y)^2. \end{aligned}$$

Now $\Re(y)$ and $\Im(y)$ are Hermitian, so that $\Re(y)^2$ and $\Im(y)^2$ are positive (this can checked by reducing to the commutative $C^*$-algebras generated by $\Re(y)$ and $\Im(y)$, respectively). It follows that

$$y^*y = 2\Re(y)^2 + 2\Im(y)^2 - yy^*$$

is also positive, proving that $y^*y = 0$ as desired. □

**Example 3.** Let $V$ be a Hilbert space and let $A = B(V)$ be the algebra of bounded operators on $V$. Then an element $f \in A$ is positive if and only if is positive when regarded as an operator: that is, if

$$(fv, v) \geq 0$$

for all $v \in V$. One direction is clear: if $f$ is positive, we can write $f = g^* g$ for some operator $g$, so that

$$(fv, v) = (g^* gv, v) = (gv, gv) = ||gv||^2 \geq 0.$$

Conversely, suppose that $f$ is positive when regarded as an operator. Then

$$(fv, v) = \overline{(fv, v)} = (v, fv) = (\overline{f}v, v)$$

for all $v \in V$, so that $f$ is Hermitian. Write $f = f_+ + f_-$ as in Proposition **??**; we wish to prove that $f_- = 0$. For every $v \in V$, we have

$$\begin{aligned} 0 &\leq (ff_-v, f_-v) \\ &= (f_-^2 v, f_-v) \\ &= (f_-^3 v, v). \end{aligned}$$

Since $f_-$ is negative, we can write $f_- = -g^2$ for some Hermitian $g$. Then

$$0 \leq (f_-^3 v, v) = -(g^6 v, v) = -||g^3(v)||,$$

so that $g^3 = 0$, hence $g = 0$ and therefore $f_- = -g^2 = 0$.

Given a pair of $C^*$-algebras $A$ and $B$, a *map of $C^*$-algebras* from $A$ to $B$ is an algebra homomorphism $f : A \to B$ satisfying $f(x^*) = f(x)^*$ for $x \in A$. Any such map is automatically has norm $\leq 1$ (and is therefore continuous): that is, we have $||f(x)|| \leq ||x||$ for $x \in A$. To prove this, we may replace $x$ by $x^* x$ and thereby reduce to the case where $x$ is Hermitian. In this case, we have

$$||f(x)|| = \rho(f(x)) = \sup\{|\lambda| : \lambda \in \sigma(f(x))\} \leq \sup\{|\lambda| : \lambda \in \sigma(x)\} = \rho(x) = ||x||$$

.

**Remark 4.** Let $f : A \to B$ be an injective map of $C^*$-algebras. Then $f$ is isometric: that is, we have $||x|| = ||f(x)||$ for all $x \in A$. To prove this, we can again reduce to the case where $x \in A$ is Hermitian. Replacing $A$ and $B$ by the sub-$C^*$-algebras generated by $x$ and $f(x)$, we can assume that $A$ and $B$ are commutative. In this case, we have seen that $f$ induces a surjective map $\operatorname{Spec} B \to \operatorname{Spec} A$, so that the pullback map $C^0(\operatorname{Spec} A) \to C^0(\operatorname{Spec} B)$ is an isometry.

**Definition 5.** Let $A$ be a $C^*$-algebra. A *representation of* $A$ is a homomorphism of $C^*$-algebras $f : A \to B(V)$, for some Hilbert space $V$.

Equivalently, a representation of $A$ is a left action of $A$ on $V$, satisfying $(x^* v, w) = (v, xw)$ for $x \in A$. Our main goal is to prove that every $C^*$-algebra $A$ admits a *faithful* representation:

**Theorem 6.** *Let $A$ be a $C^*$-algebra. Then there exists an injective map $f : A \to B(V)$, for some Hilbert space $V$.*

Since we can always take a (Hilbert space) direct sum of any collection of representations, Theorem 6 follows from the following simpler assertion:

**Proposition 7.** *Let $A$ be a $C^*$-algebra containing a nonzero element $x$. Then there exists a representation $f : A \to B(V)$ such that $f(x) \neq 0$.*

To prove Proposition 7, we need a method for constructing representations of $A$. Note that if we are given a representation of $A$ on a Hilbert space $V$ and a unit vector $v \in V$, we obtain a linear functional $\mu$ on $A$ given by

$$\mu(x) = (xv, v).$$

This linear functional has the following properties:

($a$) We have $\mu(1) = 1$ (since $v$ is a unit vector).

($b$) If $x \in A$ is Hermitian, then $\mu(x)$ is real. Equivalently, $\mu(x^*) = \overline{\mu(x)}$.

($c$) If $x \in A$ is positive, then $\mu(x) \geq 0$ (writing $x = y^*y$, we get $\mu(x) = (y^*yv, v) = (yv, yv)$ ).

($d$) The norm of $\mu$ is $\leq 1$ (in fact, it is exactly 1, since $\mu(1) = 1$).

In fact, these observations are not independent:

**Proposition 8.** *Let $A$ be a $C^*$-algebra and let $\mu : A \to \mathbf{C}$ be a continuous linear functional such that $\mu(1) = 1$. The following conditions are equivalent:*

(1) *The norm of $\mu$ is $\leq 1$.*

(2) *The function $\mu$ carries positive elements of $A$ into $\mathbb{R}_{\geq 0}$.*

(3) *The function $\mu$ arises via the above construction. That is, there exists a Hilbert space $V$, a representation of $A$ on $V$, and a unit vector $v \in V$ such that $\mu(x) = (xv, v)$.*

*Proof.* We first prove that (1) $\Rightarrow$ (2). Assume that $\mu$ satisfies (1); we first show that $\mu$ carries Hermitian elements of $A$ to real numbers. Equivalently, we will show that $\mu$ carries skew-Hermitian elements $a \in A$ to imaginary numbers. Assume otherwise; then there exists a skew-Hermitian element $a \in A$ such that $\Re(\mu(a)) \neq 0$. Replacing $a$ by $-a$ if necessary, we may suppose that $\Re(\mu(a)) > 0$. Working in the $C^*$-subalgebra of $A$ generated by $a$, we see that $||1 + \epsilon a||^2 \leq 1 + C\epsilon^2$ for some constant $C$ (where $\epsilon$ is a small real number). Then

$$(1 + \epsilon\Re(\mu(a)))^2 \leq (1 + \epsilon\Re(\mu(a)))^2 + (\epsilon\Im(\mu(a)))^2 = |\mu(1 + \epsilon(a))|^2 \leq ||1 + \epsilon(a)||^2 \leq 1 + C\epsilon^2.$$

This is impossible if $\Re(\mu(a)) > 0$, for $\epsilon$ sufficiently small.

Now let $x \in A$ be positive; we wish to prove that $\mu(x) \geq 0$. We have already seen that $\mu(x)$ is real. For small positive real numbers $\epsilon$, we have $||1 - \epsilon x|| \leq 1$ (we can check this in the commutative $C^*$-algebra generated by $x$), so that

$$1 - \epsilon\mu(x) = \mu(1) - \epsilon\mu(x) = \mu(1 - \epsilon x) \leq ||1 - \epsilon x|| \leq 1,$$

which proves that $\epsilon\mu(x) \geq 0$. Dividing by $\epsilon$, we get $\mu(x) \geq 0$.

We now prove that (2) $\Rightarrow$ (3). We define a Hermitian form $\langle , \rangle$ on $A$ by the formula $\langle x, y\rangle = \mu(y^*x)$. Since $x^*x$ is positive for $x \in A$, the quantities $\langle x, x\rangle$ are nonnegative for $x \in A$. We may therefore regard the form $\langle , \rangle$ as making $A$ into a *pre-Hilbert space*. Dividing out by the kernel of the form $\langle , \rangle$ and taking the completion, we obtain a Hilbert space $V$. For each $x \in A$, left multiplication by $x$ determines a map from $A$ to itself. Let $C = ||x||$, so that $C^2 = ||x^*x||$. Since $x^*x$ is positive, we can write $C^2 = x^*x + z^2$ for some Hermitian $z$. We have

$$\langle xy, xy\rangle = \mu(y^*x^*xy) = C^2\mu(y^*y) - \mu(y^*z^2y) \leq C^2\mu(y^*y) = C^2\langle y, y\rangle.$$

It follows that left multiplication by $x$ is bounded (by $C = ||x||$) for the norm $\sqrt{\langle y, y\rangle}$, and therefore induces a bounded operator from $V$ to itself. For $y, y' \in A$, we have

$$\langle x^*y, y'\rangle = \mu(y'^*x^*y) = \langle y, xy'\rangle,$$

and by continuity the same equation holds for $y$ and $y'$ in the completion $V$. It follows that this construction determines a representation of $A$ on $V$. By construction, the image of $1 \in A$ is a vector $v \in V$ satisfying

$$(xv, v) = \langle x1, 1 \rangle = \mu(x).$$

The implication (3) $\Rightarrow$ (1) now follows from the fact that the map $A \to B(V)$ has norm $\leq 1$ (which follows from the above calculation, and was also proven in general above). □

**Definition 9.** Let $A$ be a $C^*$-algebra. A *state* is a linear functional on $A$ satisfying the equivalent conditions of Proposition 8.

**Example 10.** Every algebra homomorphism $A \to \mathbf{C}$ is a state. In particular, every commutative $C^*$-algebra $A \simeq C^0(X)$ has plenty of states, given by evaluation at points $x \in X$. The representations given in the proof of Proposition 8 are not very interesting: the Hilbert spaces are one-dimensional.

**Proposition 11.** *Let $f : A \to B$ be an injective map of $C^*$-algebras. Then any state $\mu_0 : A \to \mathbf{C}$ can be extended to a state $\mu : B \to \mathbf{C}$.*

*Proof.* This follows from characterization (1) of Proposition 8 and the Hahn-Banach theorem. □

**Corollary 12.** *Let $A$ be a $C^*$-algebra, let $x \in A$ be a normal element, and let $\lambda \in \sigma(x)$. Then there exists a state $\mu$ of $A$ such that $\mu(x) = \lambda$.*

*Proof.* Using Proposition 11 we can replace $A$ by the $C^*$-subalgebra generated by $x$, which is commutative since $x$ is normal. The desired result now follows from Example 10. □

**Corollary 13.** *Let $A$ be a $C^*$-algebra, let $x \in A$ be a normal element, and let $\lambda \in \sigma(x)$. Then there exists a representation $V$ of $A$ containing a unit vector $v$ with $(xv, v) = \lambda$.*

*Proof of Proposition 7.* Let $x \in A$ be an arbitrary nonzero element; we wish to prove that there is a representation $V$ of $A$ on which $x$ acts nontrivially. Replacing $x$ by $x^*x$, we can assume that $x$ is normal (even positive). Then $0 < ||x|| = \rho(x)$, so there exists a nonzero element $\lambda \in \sigma(x)$. Using Corollary 13, we can find a representation $V$ and a vector $v \in V$ such that $(xv, v) = \lambda$, from which it follows that $xv \neq 0$. □