Math 261y: von Neumann Algebras (Lecture 35)

December 1, 2011

Our first goal in this lecture is to finish the example we were discussing last time. Let G be a locally compact group. Then G admits a left invariant measure μ, which is uniquely determined up to scalars (μ is called *Haar measure* on G). The measure μ need not be right invariant. For each $g \in G$, let μ^g be the measure obtained from μ by right translation by g. Then μ^g is also left invariant, so we have $\mu^g = \Delta(g)\mu$ for some scalar $\Delta(g) \in \mathbb{R}_{>0}$. The function $\Delta : G \to \mathbb{R}_{>0}$ is a group homomorphism, called the *modular function* of G. A group G is said to be *unimodular* if Δ is trivial: that is, if μ is also a right invariant measure.

Example 1. If G is discrete, there is a unique Haar measure μ on G such that every point has measure 1. Then μ^g has the same property for each $g \in G$, so that $\Delta(g) = 1$. Thus discrete groups are unimodular.

Example 2. If G is compact, there is a unique Haar measure μ on G such that $\mu(G) = 1$. Then μ^g has the same property for each $g \in G$, so that $\Delta(g) = 1$. Thus compact groups are unimodular.

Example 3. Let G be an n-dimensional real Lie group and $\mathfrak{g}$ its Lie algebra. Then the adjoint action of G on $\mathfrak{g}$ determines an action of G on $\wedge^n \mathfrak{g} \simeq \mathbb{R}$. This action is given by a character $G \to \mathbb{R}^\times$, whose absolute value is the modular function of G. Consequently, there are many non-unimodular groups: for example, the group of upper triangular matrices over $\mathbb{R}$.

Remark 4. Since right translation by gh is obtained by composing right translation by g and by h, we obtain $\Delta(gh) = \Delta(g)\Delta(h)$. That is, Δ is a group homomorphism from G to $\mathbb{R}_{>0}$. With more effort, one can show that Δ is continuous.

Remark 5. Let G be a locally compact group with Haar measure μ. Then $\Delta\mu$ is a measure on G. For each $g \in G$, we have

$$(\Delta^{-1}\mu)^g = (\Delta^{-1})^g \mu^g = (\Delta^{-1}\Delta(g)^{-1})(\Delta(g)\mu) = \Delta^{-1}\mu.$$

That is, the measure $\Delta^{-1}\mu$ is right invariant.

There is another procedure for producing a right invariant measure on G: we can take μ' to be the measure on G obtained by pulling back the measure μ along the map $g \mapsto g^{-1}$. It follows that $\mu' = c\Delta^{-1}\mu$ for some positive real number c. For any bounded open subset $U \subseteq G$ which is symmetric about the origin, we have

$$\mu(U) = \mu'(U) = c \int_U \Delta(g) d\mu.$$

That is, $\int_U (c\Delta(g)^{-1} - 1) d\mu = 0$. Taking U to be a very small neighborhood of the origin (and using the continuity of Δ) we deduce that $c = 1$: that is, we have $\mu' = \Delta^{-1}\mu$.

Let $C^0_c(G)$ denote the space of compactly supported continuous $\mathbf{C}$-valued functions on G. We regard $C^0_c(G)$ as an $*$-algebra under convolution $\star$, where

$$(f \star f')(g) = \int f(h) f'(h^{-1}g) d\mu = \int f(gh) f'(h^{-1}) d\mu.$$

$$f^*(g) = \overline{f(g^{-1})}\Delta(g^{-1}).$$

This algebra acts on $L^2(G)$ by convolution. The von Neumann algebra generated by this action is the same as the von Neumann algebra generated by *right* translations on $L^2(G)$; we denote this von Neumann algebra by $A(G)$. Let us think of $A(G)$ as a completion of $C_c^0(G)$; in particular, we will identify each element of $C_c^0(G)$ with its image in $A(G)$.

Let $\mathfrak{m}$ denote the collection of all positive elements of $A(G)$ of the form $f^* \star f$, where $f \in C_c^0(G)$. Evaluation at the identity determines a map $\phi_0 : \mathfrak{m} \to \mathbb{R}_{\geq 0}$, given by

$$f^* \star f \mapsto (f^* \star f)(e) = \int \overline{f(h^{-1})} f(h^{-1}) \Delta(h)^{-1} d\mu = \int ||f(h^{-1})||^2 d\mu' = \int ||f(h)||^2 d\mu.$$

One can show that this extends to a faithful semifinite normal weight $\phi : A(G)_+ \to \mathbb{R}_{\geq 0} \cup \{\infty\}$. The resulting inner product is well-defined on $C_c^0(G)$ and given by

$$(f, f') = (f'^* \star f)(e) = \int \overline{f'(g^{-1})} f(g^{-1}) \Delta(g^{-1}) d\mu = \int f(g) \overline{f'(g)} d\mu.$$

This induces an identification of the semicyclic representation V_ϕ with $L^2(G)$, under which the embedding $C_c^0(G) \to V_\phi$ corresponds to the inclusion of $C_c^0(G)$ into $L^2(G, \mu)$. In other words, we have $L^2(A(G)) = L^2(G, \mu)$.

Let S be the unbounded operator on $L^2(G, \mu)$ appearing in Tomita-Takesaki theory. It is given by on $C_c^0(G)$ by the formula $f \mapsto f^*$. This map is generally *not* antiunitary. If we write

$$(f, f') = \int f(g) \overline{f'(g)} d\mu,$$

then we have

$$(S(f'), S(f)) = \int f(g^{-1}) \overline{f'(g^{-1})} \Delta(g^{-1})^2 d\mu = \int f(g^{-1}) \overline{f'(g^{-1})} \Delta(g^{-1}) d\mu' = \int f(g) \overline{f'(g)} \Delta(g) d\mu.$$

where μ' is the pullback of μ along the inversion map. Consider instead the operator J given by

$$(Jf)(g) = \Delta(g)^{-1/2} \overline{f(g^{-1})}$$

We have

$$(J(f'), J(f)) = \int f(g^{-1}) \overline{f'(g^{-1})} \Delta(g)^{-1} d\mu = \int f(g^{-1}) \overline{f'(g^{-1})} d\mu' = \int f(g) \overline{f'(g)} d\mu$$

so that J is antiunitary. We have

$$S = \Delta^{-1/2} J,$$

where $\Delta^{-1/2}$ denote the unbounded operator on $L^2(G, \mu)$ given by pointwise multiplication by $\Delta^{-1/2}$. Since this is a self-adjoint operator, we see that $S = \Delta^{-1/2} J = J \Delta^{1/2}$ is the polar decomposition appearing in Tomita-Takesaki theory.

For each $g \in G$, let l_g be the unitary operator on $L^2(G, \mu)$ given by left translation by g. We also have a non-unitary operator r_g given by right translation by g. We now compute

$$\begin{aligned}
(J l_g J) f(h) &= (J l_g)(\Delta^{-1/2}(h) \overline{f(h^{-1})}) \\
&= J(\Delta^{-1/2}(gh) \overline{f(h^{-1} g^{-1})} \\
&= \Delta^{-1/2}(h) \Delta^{-1/2}(gh^{-1}) f(hg^{-1}) \\
&= \Delta^{-1/2}(g) r_{g^{-1}}(f).
\end{aligned}$$

In other words, conjugation by J carries l_g to a constant multiple of $r_{g^{-1}}$, normalized so as to be a unitary operator. It follows that the commutant of $A(G)$ in $L^2(G, \mu)$ is generated by the right translation operators r_g.

For each real number t, pointwise multiplication by Δ^{it} determines a unitary operator from $L^2(G,\mu)$ to itself. Let us denote this operator by Δ^{it}. We have

$$\begin{aligned}
(\Delta^{it} l_g \Delta^{-it}) f(h) &= (\Delta^{it} l_g) \Delta(h)^{-it} f(h) \\
&= \Delta^{it} (\Delta(gh)^{-it} f(gh) \\
&= \Delta^{it}(g) \Delta(gh)^{-it} f(gh) \\
&= \Delta(h)^{-it} l_g(f(h)).
\end{aligned}$$

That is, conjugation by Δ^{it} carries each left translation operator l_g to a scalar multiple of itself. It follows that conjugation by Δ^{it} preserves the operator algebra $A(G)$.