Math 261y: von Neumann Algebras (Lecture 33)

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Let us begin by recalling our general setup.

Notation 1. Let A be a von Neumann algebra and V a representation of A containing a cyclic and separating vector v. We let $A_{\mathbb{R}}$ denote the real vector space of self-adjoint elements of A, and $A'_{\mathbb{R}}$ the self-adjoint elements of the commutant of A. We let K denote the real Hilbert space given by the closure of the set $A_{\mathbb{R}}v \subseteq V$, and $L = iK$. Let P and Q denote the orthogonal projections onto K and L, respectively (so that P and Q are real linear operators on V). Then we have a polar decomposition

$$P - Q = J|P - Q|,$$

where $|P - Q| = (2 - P - Q)^{1/2}(P + Q)^{1/2}$. The operator $|P - Q|$ commutes with J, P and Q, while J satisfies

$$JP = (1 - Q)J \qquad JQ = (1 - P)J.$$

It follows that $J(P + Q) = (2 - P - Q)J$, and therefore $J(P + Q)^{1/2}J = (2 - P - Q)^{1/2}$. We have unitary operators $\Delta^{it} : V \to V$ for every real number t, via the formula $\Delta^{it} = (2 - P - Q)^{it}(P + Q)^{-it}$.

Let us now recall what we proved in the last lecture:

Proposition 2. *Let $x' \in A'$ and let θ be a real number with $-\pi < \theta < \pi$, so that $\Re(e^{\frac{i\theta}{2}}) > 0$. Then there exists an element $x = x_\theta \in A$ such that*

$$(P - Q)x'(P - Q) = e^{i\frac{\theta}{2}}(2 - P - Q)x(P + Q) + e^{-i\frac{\theta}{2}}(P + Q)x_\theta(2 - P - Q).$$

In this lecture, we will establish the following relationship between x_θ and x':

Proposition 3. *We have*

$$x_\theta = \frac{1}{2}\int \frac{e^{-\theta t}}{\cosh(\pi t)}\Delta^{it}Jx'J\Delta^{-it}dt.$$

Remark 4. Here we have made use of integration in the Banach space $B(V)$ of all bounded operators on V. More concretely, the formula of Proposition has the following interpretation: for every pair of vectors $w, w' \in V$, we have an equality of complex numbers

$$(x_\theta w, w') = \frac{1}{2}\int \frac{e^{-\theta t}}{\cosh(\pi t)}(\Delta^{it}Jx'J\Delta^{-it}w, w')dt.$$

Let us assume Proposition 3 and use it to complete the proof of our main result:

Corollary 5. *Let $x' \in A'$. Then, for every real number t, the operator $\Delta^{it}Jx'J\Delta^{-it}$ belongs to A.*

Proof. Since $A = A''$, it will suffice to show that each of the operators $\Delta^{it}Jx'J\Delta^{-it}$ commutes with each operator in A'. Fix $y' \in A'$. For every real number t, let $F(t)$ denote the operator given by the commutator of y' with $\Delta^{it}Jx'J\Delta^{-it}$. We wish to prove that $F(t) = 0$ for all t. Fix $w, w' \in V$, and set $f(t) = (F(t)w, w')$;

we wish to prove that $f(t)=0$ for all $t\in\mathbb{R}$. Note that the function $t\mapsto f(t)$ is bounded and continuous, so that the functor $t\mapsto \frac{f(t)}{\cosh(\pi t)}$ decays as fast as $e^{-\pi t}$ as $t\to\infty$ and $e^{\pi t}$ as $t\to-\infty$. It follows that for any z with $-\pi<\Re(z)<\pi$, the integral

$$g(z)=\int e^{-zt}\frac{f(t)}{\cosh(\pi t)}dt$$

is well-defined, and depends holomorphically on z. Using Proposition 3, we see that $g(\theta)=0$ for $-\pi<\theta<\pi$. It follows that $g=0$. In particular, g vanishes on the imaginary axis, so that the Fourier transform of $\frac{f(t)}{\cosh(\pi t)}$ vanishes. It follows that $f(t)=0$ for all $t\in\mathbb{R}$. □

We now turn to the proof of Proposition 3. We will henceforth regard θ as fixed, and simply write x for x_θ. We wish to prove that

$$x=\frac{1}{2}\int\frac{e^{-\theta t}}{\cosh(\pi t)}\Delta^{it}Jx'J\Delta^{-it}dt.$$

Since the operator $|P-Q|=(P+Q)^{1/2}(2-P-Q)^{1/2}$ is injective, it will suffice to prove this equality after multiplying both sides by $|P-Q|$ on the left and the right. Using the fact that $|P-Q|$ commutes with Δ^{it} and $|P-Q|J=P-Q$, we can rewrite the desired equality as

$$|P-Q|x|P-Q|=\frac{1}{2}\int\frac{e^{-\theta t}}{\cosh(\pi t)}\Delta^{it}(P-Q)x'(P-Q)\Delta^{-it}dt.$$

Using the relationship between x and x', we see that the right hand side is given by

$$\frac{1}{2}\int\frac{e^{-\theta t}}{\cosh(\pi t)}(e^{i\frac{\theta}{2}}\Delta^{it}(2-P-Q)x(P+Q)\Delta^{-it}+e^{-i\frac{\theta}{2}}\Delta^{it}(P+Q)x(2-P-Q)\Delta^{-it})dt$$

Using the definition of Δ^{it}, we can write this as

$$\frac{1}{2}\int\frac{e^{-\theta t}}{\cosh(\pi t)}(e^{i\frac{\theta}{2}}(2-P-Q)^{1+it}(P+Q)^{-it}x(P+Q)^{1+it}(2-P-Q)^{-it}+e^{-i\frac{\theta}{2}}(2-P-Q)^{it}(P+Q)^{1-it}x(P+Q)^{it}(2-P-Q)^{1-it})dt$$

Meanwhile, the left hand side can be written as

$$(P+Q)^{1/2}(2-P-Q)^{1/2}x(P+Q)^{1/2}(2-P-Q)^{1/2}.$$

To compare these, let us consider, for each complex number z with $\frac{-1}{2}\leq\Re(z)\leq\frac{1}{2}$, the operator

$$F(z)=(P+Q)^{1/2-z}(2-P-Q)^{1/2+z}x(P+Q)^{1/2+z}(2-P-Q)^{1/2-z}.$$

Then the desired inequality can be rewritten as

$$F(0)=\frac{1}{2}\int\frac{e^{-\theta t}}{\cosh(\pi t)}(e^{i\frac{\theta}{2}}F(it+\frac{1}{2})+e^{-i\frac{\theta}{2}}F(it-\frac{1}{2}))dt.$$

Fix vectors $w,w'\in V$, and define $f(z)=(F(z)w,w')$. We then wish to prove an equality of complex numbers

$$f(0)=\frac{1}{2}\int\frac{e^{-\theta t}}{\cosh(\pi t)}(e^{i\frac{\theta}{2}}f(it+\frac{1}{2})+e^{-i\frac{\theta}{2}}f(it-\frac{1}{2}))dt.$$

Note that the function f is bounded on the set $\{z\in\mathbf{C}:\frac{-1}{2}\leq\Re(z)\leq\frac{1}{2}\}$, and holomorphic on the interior of this region. We define a function g on the same region by the formula

$$g(z)=\frac{\pi e^{i\theta z}f(z)}{\sin(\pi z)}.$$

Then $g(z)$ has a simple pole at the origin. Moreover the numerator of $g(z)$ is bounded in absolute value by some constant multiple of $e^{-\theta\Im(z)}$, while the denominator grows like $e^{\pi|\Im(z)|}$ as $\Im(z) \to \pm\infty$. Since $-\pi < \theta < \pi$, it follows that differential $g(z)$ decays exponentially in $|\Im(z)|$. Applying the Cauchy integral formula to rectangles bounded by the vertices $\pm\frac{1}{2} \pm it$ and taking the limit as $t \to \infty$, we obtain the formula

$$\begin{aligned}
f(0) &= \mathrm{Res}_0(g) \\
&= \frac{1}{2\pi i}(i\int g(\frac{1}{2}+it)dt - i\int g(it-\frac{1}{2})dt) \\
&= \frac{1}{2}\int \frac{e^{i\theta(\frac{1}{2}+it)}}{\sin(\frac{\pi}{2}+i\pi t)} f(it+\frac{1}{2}) - \frac{e^{i\theta}(it-\frac{1}{2})}{\sin(i\pi t-\frac{\pi}{2})} f(it-\frac{1}{2})dt \\
&= \frac{1}{2}\int e^{i\frac{\theta}{2}}\frac{e^{-\theta t}}{\cosh(\pi t)} f(it+\frac{1}{2}) - e^{-i\frac{\theta}{2}}\frac{e^{-\theta t}}{-\cosh(\pi t)} f(it-\frac{1}{2})dt \\
&= \frac{1}{2}\int e^{i\frac{\theta}{2}}\frac{e^{-\theta t}}{\cosh(\pi t)} f(it+\frac{1}{2}) + e^{-i\frac{\theta}{2}}\frac{e^{-\theta t}}{\cosh(\pi t)} f(it-\frac{1}{2})dt.
\end{aligned}$$

as desired.