Math 261y: von Neumann Algebras (Lecture 32)

November 18, 2011

Let us recall the setting of the last lecture.

Notation 1. Let A be a von Neumann algebra and V a representation of A containing a cyclic and separating vector v. We let $A_{\mathbb{R}}$ denote the real vector space of self-adjoint elements of A, and $A'_{\mathbb{R}}$ the self-adjoint elements of the commutant of A. We let K denote the real Hilbert space given by the closure of the set $A_{\mathbb{R}}v \subseteq V$, and $L = iK$. Let P and Q denote the orthogonal projections onto K and L, respectively (so that P and Q are real linear operators on V). Then we have a polar decomposition

$$P - Q = J|P - Q|,$$

where $|P - Q| = (2 - P - Q)^{1/2}(P + Q)^{1/2}$. The operator $|P - Q|$ commutes with J, P and Q, while J satisfies

$$JP = (1 - Q)J \qquad JQ = (1 - P)J.$$

It follows that $J(P+Q) = (2-P-Q)J$, and therefore $J(P+Q)^{1/2}J = (2-P-Q)^{1/2}$. In the last lecture, we defined unitary operators $\Delta^{it} : V \to V$ for every real number t, via the formula $\Delta^{it} = (2-P-Q)^{it}(P+Q)^{-it}$.

In this lecture we begin the proof of the following result (which implies the main results of Tomita-Takesaki theory, as we saw last time:

Proposition 2. *For every real number t, we have $J\Delta^{it}A'\Delta^{-it}J \subseteq A$.*

Proposition 2 implies in particular that to every element $x' \in A'$, we can associate many different elements of A (one for each real number t). The key to proving Proposition 2 is to find these elements. The idea of the proof is not to look for them individually (that is, not to try to prove the result for any individual value of t), but instead to look for an appropriate linear combination of them. We begin with the following:

Proposition 3. *Let $x' \in A'_{\mathbb{R}}$ and let λ be a complex number with $\Re(\lambda) > 0$. Then there exists an element $x \in A_{\mathbb{R}}$ such that*

$$(x'v, yv) = \Re(\lambda(xv, yv))$$

for all $y \in A_{\mathbb{R}}$.

Remark 4. Note that if $x' \in A'_{\mathbb{R}}$, then for all $y \in A_{\mathbb{R}}$ we have

$$(x'v, yv) = (yx'v, v) = (x'yv, v) = (yv, x'v)$$

so that $(x'v, yv)$ is automatically real. It follows that $x'v$ is orthogonal to $iA_{\mathbb{R}}v$ if we view V is a real Hilbert space: that is, $Qx'v = 0$.

Proof. We may assume without loss of generality that $\Re(\lambda) = 1$ and $||x'|| = 1$. Let $\phi : A_{\mathbb{R}} \to \mathbb{R}$ be the linear functional given by $\phi(y) = (x'v, yv)$, and for each $x \in A_{\mathbb{R}}$ let $\phi_x : A_{\mathbb{R}} \to \mathbb{R}$ be given by $\phi_x(y) = \Re(\lambda(xv, yv))$. We will prove that $\phi = \phi_x$ for some x lying in the unit ball of $A_{\mathbb{R}}$.

The map

$$x \mapsto \phi_x$$

is continuous, where we endow $A_{\mathbb{R}}$ with the weak operator topology and $A_{\mathbb{R}}^{\vee}$ with the weak $*$-topology. Since the unit ball of $A_{\mathbb{R}}$ is compact with respect to the weak operator topology, its image $Z = \{\phi_x : ||x|| \leq 1\}$ is compact. In particular, Z is closed. The set Z is also convex. Consequently, if $\phi \notin Z$, then there exists a weak $*$-continuous functional on $A_{\mathbb{R}}^{\vee}$ which separates Z from ϕ. Such a functional is necessarily given by evaluation on some element $y \in A_{\mathbb{R}}$. That is, we can choose an element $y \in A_{\mathbb{R}}$ such that $\Re(\lambda(xv, yv)) \leq 1$ for every x in the unit ball of $A_{\mathbb{R}}$, while $(x'v, yv) > 1$. Let $y = u|y|$ be the polar decomposition of y. Since $y \in A_{\mathbb{R}}$, the operators u and $|y|$ belong to $A_{\mathbb{R}}$ and commute with one another, and with the operator x'. Since $||x'|| \leq 1$, we have $x'y \leq |y|$ (since we can check this in the commutative case).

We now have

$$\begin{aligned} 1 &< (x'v, yv) \\ &= (v, x'yv)) \\ &\leq (v, |y|v) \\ &= (uv, yv). \end{aligned}$$

Since u belongs to the unit ball of $A_{\mathbb{R}}$, we have $(uv, yv) \leq 1$, a contradiction. □

Proposition 3 is particularly interesting in the case $\lambda = 1$. In this case, we have $(x'v, yv) = \Re(xv, yv)$ for all $y \in A_{\mathbb{R}}$. In other words, xv is given by taking the orthogonal projection P and applying it to the element $x'v$. We have already observed that $Qx'v = 0$. It follows that x is given by $(P-Q)x'v$. This proves a special case of the following:

Corollary 5. *For each $x' \in A'$, there exists an element $x \in A$ such that*

$$(P - Q)x'v = xv \qquad (P - Q)x'^{*}v = x^{*}v.$$

Proof. When $x' \in A'_{\mathbb{R}}$, this follows from the arguments given above. In general, we can write $x' = x'_0 + ix'_1$ where x'_0 and x'_1 are self-adjoint. Applying the result in the adjoint case, we obtain elements $x_0, x_1 \in A_{\mathbb{R}}$ with $(P - Q)x'_0 v = x_0 v$ and $(P - Q)x'_1 v = x_1 v$. Then $x = x_0 - ix_1$ has the desired properties. □

Proposition 6. *Let $x' \in A'$, and let $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then there exists an element $x \in A$ such that*

$$(P - Q)x'(P - Q) = e^{i\theta}(2 - P - Q)x(P + Q) + e^{-i\theta}(P + Q)x(2 - P - Q).$$

Proof. Without loss of generality, we may assume that x' is self-adjoint. Take $\lambda = 2e^{i\theta}$, and let $x \in A_{\mathbb{R}}$ be the element given in Proposition 3, so that

$$(x'v, yv) = 2\Re(e^{i\theta}(xv, yv)) = e^{i\theta}(xv, yv) + e^{-i\theta}(yv, xv) = e^{i\theta}(xv, yv) + e^{-i\theta}(y^{*}v, xv)$$

for all $y \in A_{\mathbb{R}}$. Since the left and right hand sides are both conjugate linear in y, we have

$$(x'v, yv) = e^{i\theta}(xv, yv) + e^{-i\theta}(y^{*}v, xv) \tag{1}$$

for all $y \in A_{\mathbb{R}}$. We will prove that x has the desired property.

We first replace y by $z^{*}y$ in equation 1 to obtain

$$(x'v, z^{*}yv) = e^{i\theta}(xv, z^{*}yv) + e^{-i\theta}(y^{*}zv, xv)$$

$$(x'zv, yv) = (zx'v, yv) = e^{i\theta}(zxv, yv) + e^{-i\theta}(zv, yxv).$$

Now suppose that $y', z' \in A'$, and take y and z to be the (uniquely determined) elements of A satisfying $yv = (P - Q)y'v$, $zv = (P - Q)z'v$. We then obtain

$$(x'(P - Q)z'v, (P - Q)y'v) = e^{i\theta}(zxv, (P - Q)y'v) + e^{-i\theta}((P - Q)z'v, yxv).$$

We can rewrite the left hand side as

$$(y'v, (P-Q)x'(P-Q)z'v).$$

The right hand side is given by

$$e^{i\theta}(y'v, (P-Q)zxv) + e^{-i\theta}((P-Q)yxv, z'v).$$

Note that if $f \in A$ is self-adjoint, then $fv \in K$ so that $P(fv) = fv$. It follows that $(P-Q)(fv) = (2-P-Q)(fv) = (2-P-Q)(f^*v)$. Since $(P-Q)(fv)$ and $(2-P-Q)(f^*v)$ are both conjugate linear in f, we conclude that $(P-Q)(fv) = (2-P-Q)f^*v$ for all $f \in A$. In particular, we have

$$(y'v, (P-Q)zxv) = (y'v, (2-P-Q)x^*z^*v) = (y'v, (2-P-Q)x(P-Q)z'^*v).$$

If $g \in A'_{\mathbb{R}}$ is self-adjoint, then $gv \in L^\perp$ so that $Q(gv) = 0$, and therefore $(P-Q)(gv) = (P+Q)(gv) = (P+Q)(g^*v)$. The same argument shows that the equality $(P-Q)(gv) = (P+Q)(g^*v)$ is true for all $g \in A'$. It follows that $(y'v, (P-Q)zxv) = (y'v, (2-P-Q)x(P-Q)z'^*v) = (y'v, (2-P-Q)x(P+Q)z'v)$. Similarly, we have

$$((P-Q)yxv, z'v) = ((2-P-Q)xy^*v, z'v) = ((2-P-Q)x(P-Q)y'^*v, z'v) = ((2-P-Q)x(P+Q)y'v, z'v).$$

We therefore obtain

$$(y'v, (P-Q)x'(P-Q)z'v) = e^{i\theta}(y'v, (2-P-Q)x(P+Q)z'v) + e^{-i\theta}(y'v, (P+Q)x(2-P-Q)z'v).$$

Since v is a separating vector, $A'v$ is dense in V. It follows that

$$(P-Q)x'(P-Q) = e^{i\theta}(2-P-Q)x(P+Q) + e^{-i\theta}(P+Q)x(2-P-Q),$$

as desired. □