Math 261y: von Neumann Algebras (Lecture 31)

November 16, 2011

Let us recall the setting of the last lecture.

Notation 1. Let A be a von Neumann algebra and V a representation of A containing a cyclic and separating vector v. We let K denote the real Hilbert space given by the closure of the set $\{xv : x^* = x\}$, and L the real Hilbert space given by the closure of the set $\{xv : x^* = -x\}$. Let P and Q denote the orthogonal projections onto K and L, respectively (so that P and Q are real linear operators on V). Then we have a polar decomposition

$$P - Q = J|P - Q|,$$

where $|P - Q| = (2 - P - Q)^{1/2}(P + Q)^{1/2}$. The operator $|P - Q|$ commutes with J, P and Q, while J satisfies

$$JP = (1 - Q)J \qquad JQ = (1 - P)J.$$

It follows that $J(P + Q) = (2 - P - Q)J$, and therefore $J(P + Q)^{1/2}J = (2 - P - Q)^{1/2}$.

Note that in the present setting, V is a complex Hilbert space and we have $L = iK$. It follows that $Qi = iP$. In particular, $i(P + Q) = iP + iQ = Qi + Pi = (P + Q)i$, so that $P + Q$ is a positive $\mathbf{C}$-linear operator.

Construction 2. Let $T : V \to V$ be a normal $\mathbf{C}$-linear bounded operator and let $\sigma(T)$ be its spectrum. Recall that for every bounded Borel measurable function f, we can define a new operator $f(T) \in B(V)$ using the Borel functional calculus. We will be particularly interested in the case where T is a positive operator, so that σ is a bounded subset of $\mathbb{R}_{\geq 0}$. For every complex number z with $\Re(z) \geq 0$, we can take f to be the function

$$t \mapsto t^z = \begin{cases} 0 & \text{if } t = 0 \\ e^{z \log(t)} & \text{if } t > 0. \end{cases}$$

(Note that this function is bounded on bounded subsets of $\mathbb{R}$.) In this case, we will denote the operator $f(T)$ by T^z. It belongs to every von Neumann subalgebra of $B(V)$ which contains x, and therefore commutes with every operator which commutes with x. Since the functional calculus is multiplicative, we have $T^{z+z'} = T^z T^{z'}$ for $z, z' \in \mathbf{C}$ with $\Re(z), \Re(z') \geq 0$. Note that T^0 is the projection operator which annihilates $\ker(T)$; in particular, $T^0 = 1$ if T is injective.

Since $f(T)^* = \overline{f}(T^*)$, we see that $(T^z)^* = T^{\overline{z}}$ when T is a positive operator. In particular, T^z is self-adjoint when z is real. If T is injective, then T^{it} is inverse to $T^{-it} = (T^{it})^*$ for $t \in \mathbb{R}$, so that the operators T^{it} are unitary.

If $0 \neq v \in V$ is an eigenvector for T with eigenvalue λ (that is, if $Tv = \lambda v$), then the positivity of T implies that $\lambda \geq 0$. If T is injective, we even have $\lambda > 0$. If $\Re(z) \geq 0$, we have $T^z(v) = \lambda^z v$.

Let us now return to the situation of interest: V is a representation of A with a cyclic and separating vector v. Now $P + Q$ and $2 - P - Q$ are injective positive self-adjoint operators. We can therefore define complex powers $(P + Q)^z$ and $(2 - P - Q)^z$ for $\Re(z) \geq 0$. Note that when $z = \frac{1}{2}$, this agrees with our earlier definition (that is, we obtain the unique positive square roots of $P+Q$ and $2-P-Q$). Since J is an antiunitary map satisfying

$$J(P+Q)J^{-1} = 2-P-Q,$$

we deduce that

$$J(P+Q)^z J^{-1} = (2-P-Q)^{\overline{z}}$$

for every complex number z satisfying $\Re(z) \geq 0$. We are primarily interested in the case where z is purely imaginary. In this case, we obtain

$$J(P+Q)^{it} J^{-1} = (2-P-Q)^{-it}$$

or $J(P+Q)^{it} = (2-P-Q)^{-it} J$.

Definition 3. For every real number t, we let Δ^{it} denote the unitary operator given by $(2-P-Q)^{it}(P+Q)^{-it}$.

Remark 4. Recall that the unbounded operator $S : V \to V$ given by the closure of the operator $xv \mapsto x^* v$ has a polar decomposition $S = J\Delta^{1/2}$, where $\Delta^{1/2} = (2-P-Q)^{1/2}(P+Q)^{-1/2}$. Thus Definition 3 is at least morally consistent with our earlier notation. In fact, one can extend Construction 2 to define complex powers of possible unbounded positive self-adjoint operators, so that the unitary operators Δ^{it} can be obtained directly from the unbounded operator $\Delta^{1/2}$.

Remark 5. Conjugation by J carries $(P+Q)^{it}$ to $(2-P-Q)^{-it}$. It follows that J commutes with Δ^{it} for every real number t.

We can now state more fully the main results of Tomita-Takesaki theory.

Theorem 6. *Let A be a von Neumann algebra, let V be a representation of A containing a cyclic and separating vector v, and define J and Δ^{it} as above. Then:*

(1) *We have $A' = JAJ$.*

(2) *For every real number t, conjugation by Δ^{it} preserves the von Neumann algebras A and A'.*

(3) *If c belongs to the center of A, then $JcJ = c^*$.*

Remark 7. In the situation of Theorem **??**, we get a one-parameter group of automorphisms $\{\sigma_t\}_{t \in \mathbb{R}}$ of A, given by $\sigma_t(x) = \Delta^{it} x \Delta^{-it}$. This family depends on the choice of (V, v). Since the pair (V, v) can be reconstructed from the state $\phi(x) = (xv, v)$, the construction $t \mapsto \sigma_t$ is called the *modular flow* associated to the state given by $\phi(x) = (xv, v)$.

The core of Theorem 6 is contained in the following result:

Proposition 8. *In the situation of Theorem 6, for every real number t, conjugation by the operator $J\Delta^{it}$ carries A' into A.*

We will begin the proof of Proposition 8 in the next lecture. Let us assume Proposition 8 for the time being and explain how it leads to a proof of the whole of Theorem 6.

Proof of Theorem 6. Taking $t = 0$ in Proposition 8, we see that $JA'J \subseteq A$. To prove (1), it suffices to verify the reverse inclusion. Equivalently, we must show that $JAJ \subseteq A'$.

Note that the vector v belongs to the real Hilbert space K, so that $Pv = v$. If $x \in A$ is skew-adjoint, then we have $(xv, v) = (v, x^* v) = (v, -xv) = -\overline{(xv, v)}$. That is, the complex number (xv, v) is purely imaginary. It follows that $v \in L^{\perp}$, so that $Qv = 0$. Thus $(P+Q)v = v$ and $(2-P-Q)v = v$. It follows that v is fixed by $(P+Q)^{1/2}$ and $(2-P-Q)^{1/2}$, and therefore also by $|P-Q| = (P+Q)^{1/2}(2-P-Q)^{1/2}$. Since v is also fixed by $J|P-Q| = P-Q$, we deduce that $Jv = v$.

Let $A_{\mathbb{R}}$ denote the set of self-adjoint elements of A. Since $J(K) = L^{\perp}$, the complex number (Jxv, yv) is real for every pair of elements $x, y \in A_{\mathbb{R}}$. We therefore have

$$(Jxv, yv) = (yv, Jxv) = (xv, Jyv)$$

so that $(yJxv, v) = (v, xJyv)$. Both sides of this equation are $\mathbf{C}$ linear functions of y and and $\mathbf{C}$-antilinear in x, so the identity holds for all $x, y \in A$.

Let $z' \in A'$. Using Proposition 8, we deduce that $Jz'J \in A$. Replacing y by $yJz'J$ in the above identity, we get

$$(yJz'JJxv, v) = (v, xJyJz'Jv)$$

or $(yJz'xv, v) = (v, xJyJz'v)$. Since z' commutes with x, the left hand side is given by

$$\begin{aligned} (yJz'xv, v) &= (yJxz'v, v) \\ &= (Jxz'v, y^*v) \\ &= (Jy^*v, xz'v) \\ &= (x^*Jy^*v, z'v) \\ &= (x^*(Jy^*J)v, z'v). \end{aligned}$$

Similarly, the right hand side is given by

$$\begin{aligned} (v, xJyJz'v) &= (x^*v, JyJz'v) \\ &= (yJz'v, Jx^*v) \\ &= (Jz'v, y^*Jx^*v) \\ &= (Jy^*Jx^*v, z'v) \\ &= ((Jy^*J)x^*v, z'v). \end{aligned}$$

Since v is a separating vector, $A'v$ is dense in V. Hence the equalities

$$(x^*Jy^*Jv, z'v) = ((Jy^*J)x^*v, z'v)$$

imply that $x^*(Jy^*J)v = (Jy^*J)x^*v$ for all $x, y \in A$. Replacing x^* by xz and y^* by y, we obtain

$$xz(JyJ)v = (JyJ)xzv.$$

The same identity gives $z(JyJ)v = (JyJ)zv$, so we can rewrite our equality as

$$x(JyJ)zv = (JyJ)xzv.$$

Since v is a cyclic vector, Av is dense in V. It follows that $x(JyJ) = (JyJ)x$ for all $x, y \in A$. This completes the proof of (1).

To prove (2), we note that for every real number t we have

$$\Delta^{it} A \Delta^{-it} = \Delta^{it} J A' J \Delta^{-it} \subseteq A$$

by virtue of Proposition 8. The reverse inequality follows by replacing t with $-t$.

We now prove (3). It is easy to see that the collection of those elements $c \in Z(A)$ which satisfy $JcJ = c^*$ is a $\mathbf{C}$-vector space which is closed in the norm topology. It will therefore suffice to show that $JcJ = c^*$ in the case where c is a central projection of A. In this case, we can decompose A as a product $A_- \times A_+$ and V as a product $V_- \times V_+$, so that c is given by orthogonal projection onto V_-. Unwinding the definitions, we note that J decomposes into a pair of antiunitary involutions on V_- and V_+. In particular, J commutes with the orthogonal projection onto V_-, so that $JcJ = c = c^*$. □