Math 261y: von Neumann Algebras (Lecture 30)

November 13, 2011

In the last lecture, we asserted the following without proof:

Theorem 1. *Let A be a von Neumann algebra, V a representation of A with a cyclic and separating vector v, and let $S = J\Delta^{\frac{1}{2}}$ be the closure of the operator $xv \mapsto x^*v$. Then:*

(1) *We have $A' = JAJ$. That is, conjugation by J induces a conjugate-linear isomorphism of A with its commutant A'.*

(2) *If $z \in Z(A)$, then $Jz = z^*J$.*

Our goal in this lecture is to begin the proof of Theorem 1. Our first step is to better understand the operator $\Delta^{\frac{1}{2}}$. We have already seen that if $T : V \to V$ is a positive densely defined injective self-adjoint operator, then T has the form GF^{-1}, where $F, G : V \to V$ are positive bounded injective operators satisfying $F^2 + G^2 = 1$. Actually, in what follows, it will be convenient to have a slight variation on this statement (obtained by multiplying F and G by $\sqrt{2}$):

Lemma 2. *Let $T : V \to V$ be a positive, densely defined, injective, self-adjoint unbounded operator from a Hilbert space V to itself. Then T can be written uniquely in the form GF^{-1}, where $F, G : V \to V$ are positive bounded injective operators satisfying $F^2 + G^2 = 1$. Conversely, for every such pair (F, G), the map GF^{-1} is a densely defined positive self-adjoint unbounded operator on V.*

Our next goal is to try to locate the operators F and G in the situation of Theorem 1. In what follows, we will think of V as a *real* Hilbert space. Let $K_0 \subseteq V$ denote the real subspace consisting of all vectors of the form xv, where $x \in A$ is self-adjoint, and let $K \subseteq V$ denote the closure of K. Similarly, we let $L_0 \subseteq V$ denote the real subspace consisting of all vectors of the form xv where $x \in A$ is skew-adjoint, and L its closure. Then K and L are real Hilbert spaces, and we have $K = iL$.

Proposition 3. *In the above situation, the domain of the operator S is the space $K + L$. Moreover, we have $S(w + w') = w - w'$ for $w \in K$ and $w' \in L$.*

Proof. If $w \in K_0$, then w belongs to the domain of S and satisfies $S(w) = w$. Since S is closed, we see that $K \subseteq \mathrm{Dom}(S)$ and S acts by the identity on K. Similarly, $L \subseteq \mathrm{Dom}(S)$ and S acts by -1 on L. This proves that $K + L \subseteq \mathrm{Dom}(S)$. Conversely, suppose that $y \in \mathrm{Dom}(S)$. Then there exists a sequence of elements $x_i \in A$ such that $x_i v \to y$ and $x_i^* v \to S(y)$. Then $(x_i + x_i^*)v \to y + S(y)$, so that $y + S(y) \in K$. Similarly, $y - S(y) \in L$. It follows that $y \in K + L$. □

Since S is well-defined, we see that $K \cap L = 0$. Moreovrer, $K + L = \mathrm{Dom}(S)$ is dense in V. Let us now abstract the essence of the above situation.

Construction 4. Let V be a real Hilbert space containing closed subspaces K and L such that $K \cap L = 0$ and $K + L$ is dense in V. We define an unbounded operator $S : V \to V$ with domain $K + L$ by the formula $S(w + w') = w - w'$ for $w, w' \in V$.

Equivalently, we define S so that the graph Γ of S is given by

$$\{(u, u') \in V \times V : u + u' \in K, u - u' \in L\}.$$

From this description it follows immediately that S is a closed operator.

In the setting of Construction 4, we would like to describe the polar decomposition of S. To this end, let P and Q denote the operators given by orthogonal projection onto K and L, respectively. The bounded operator $P - Q$ has a polar decomposition

$$P - Q = J|P - Q|.$$

We will show that J coincides with the operator appearing in Theorem 1: that is, that $J^{-1}S$ is a positive self-adjoint unbounded operator.

We first note that since P and Q are projections, we have $0 \leq P + Q \leq 2$. In particular, $P + Q$ and $2 - P - Q$ have unique positive square roots, which we will denote by $(P + Q)^{1/2}$ and $(2 - P - Q)^{1/2}$. We first prove:

Lemma 5. *In the above situation, we have* $|P - Q| = (2 - P - Q)^{1/2}(P + Q)^{1/2}$.

Proof. Since both sides are positive self-adjoint operators, it will suffice to prove that the identity holds after squaring both sides. That is, we wish to show

$$(2 - P - Q)(P + Q) = |P - Q|^2 = (P - Q)^2.$$

This follows by expanding both sides, using the fact that P and Q are projections. □

Since $P - Q$ is self-adjoint, we have

$$J|P - Q| = P - Q = (P - Q)^* = (J|P - Q|)^* = |P - Q|J^{-1} = J^{-1}(J|P - Q|J^{-1}).$$

From the uniqueness of the polar decomposition, we conclude that $J = J^{-1}$ and that J commutes with $|P - Q|$. It follows that J commutes with $J|P - Q| = P - Q$.

Proposition 6. *The operators* $P+Q$, $(2-P-Q)$, *and* $|P-Q|$ *are injective. The operator* $|P-Q|$ *commutes with* P *and* Q. *Moreover, we have*

$$JP = (1 - Q)J \qquad JQ = (1 - P)J.$$

Proof. If $v \in V$ satisfies $(P + Q)v = 0$, then

$$0 = ((P + Q)v, v) = (Pv, v) + (Qv, v) = ||Pv||^2 + ||Qv||^2$$

so that v is orthogonal to both K and L. Since $K + L$ is dense, we conclude that $v = 0$. This proves that $P + Q$ is injective. To show that $2 - P - Q$ is injective, we can apply the same argument to the pair of subspaces $K^{\perp}, L^{\perp} \subseteq V$. Since $|P - Q| = (P + Q)^{1/2}(2 - P - Q)^{1/2}$, we conclude that $|P - Q|$ is injective.

To prove that $|P - Q|$ commutes with P and Q, it suffices to show that $|P - Q|^2 = (P - Q)^2 = P + Q - PQ - QP$ commutes with P and Q. An easy calculation gives

$$P(P + Q - PQ - QP) = P - PQP = (P + Q - PQ - QP)P,$$

so that $|P - Q|$ commutes with P; the proof for Q is similar.

We now prove that $JP = (1 - Q)J$ (the proof that $JQ = (1 - P)J$ is similar). We have

$$|P - Q|JP = (P - Q)P = P - QP = (1 - Q)(P - Q) = (1 - Q)|P - Q|J = |P - Q|(1 - Q)J.$$

Since $|P - Q|$ is injective we obtain $JP = (1 - Q)J$. □

Corollary 7. *We have* $J(K) = L^{\perp}$ *and* $J(L) = K^{\perp}$.

Corollary 8. *The adjoint of the unbounded operator* S *is given by* JSJ.

Proof. Note that JSJ is an unbounded operator with domain $K^\perp + L^\perp$, given by

$$(JSJ)(v+w) = -v + w$$

for $v \in K^\perp$ and $w \in L^\perp$. If $v' \in K$ and $w' \in L$, we have

$$(JSJ(v+w), v'+w') = (-v+w, v'+w') = (w, v') - (v, w') = (v+w, v-w') = (v+w, S(v'+w')).$$

This proves that $JSJ \subseteq S^*$. Conversely, suppose that u belongs to the domain of S^*. Then for $v' \in K, w' \in L$ we have

$$(S^*u, v'+w') = (u, v'-w').$$

Taking $w' = 0$, we deduce that $S^*(u) - u \in K^\perp$. Taking $v' = 0$, we deduce that $S^*(u) + u \in L^\perp$. It follows that $u \in K^\perp + L^\perp$, so that the domain of S^* is contained in $K^\perp + L^\perp$. This immediately implies that $S^* = JSJ$. □

Corollary 9. *The operator JS is self-adjoint.*

This is not quite enough to prove that $S = J(JS)$ is the polar decomposition of S: we also need to know that JS is a positive unbounded operator. We can deduce this from the following more precise result:

Proposition 10. *The operator S is given by $J(2-P-Q)^{1/2}(P+Q)^{-1/2}$.*

Proof. It follows from Lemma 2 that $T = (2-P-Q)^{1/2}(P+Q)^{-1/2}$ is a self-adjoint unbounded operator from V to itself. We will prove that $JT \subseteq S$. It then follows that $JSJ = S^* \subseteq (JT)^* = TJ$. Conjugating by J, we get $S \subseteq JT$, so that $JT = S$.

We have seen that $P+Q$ is a injective operator. It follows that $(P+Q)^{1/2}$ is also injective. Since this operator is self adjoint, it has dense image. It follows that the graph of the operator $(2-P-Q)^{1/2}(P+Q)^{1/2}(P+Q)^{-1}$ is dense in the graph of T. That is, T is the closure of the operator $|P-Q|(P+Q)^{-1}$, so that JT is the closure of $J|P-Q|(P+Q)^{-1} = (P-Q)(P+Q)^{-1}$. It will therefore suffice to show that $(P-Q)(P+Q)^{-1} \subseteq S$. This is clear: if u belongs to the domain of $(P+Q)^{-1}$, then we can write $u = (P+Q)v$ for some $v \in V$, so that

$$S(u) = S(Pv + Qv) = Pv - Qv = (P-Q)v = (P-Q)(P+Q)^{-1}(u).$$

□