Math 261y: von Neumann Algebras (Lecture 3)

September 7, 2011

In this lecture, we continue our study of C^*-algebras. Recall that C^*-algebra is a Banach algebra equipped with an anti-involution $x \mapsto x^*$ satisfying

$$||x||^2 = ||x^* x||.$$

Notation 1. Let A be a $*$-algebra. We say that an element $x \in A$ is *Hermitian* or *self-adjoint* if $x = x^*$. We say that x is *skew-Hermitian* or *skew-adjoint* if $x^* = -x$. Every element $x \in A$ admits a unique decomposition $x = \Re(x) + i\Im(x)$, where $\Re(x) = \frac{x+x^*}{2}$ is self-adjoint and $i\Im(x) = \frac{x-x^*}{2}$ is skew-adjoint.

We say that an element $x \in A$ is *normal* if x and x^* commute: equivalently, x is normal if $\Re(x)$ and $\Im(x)$ commute.

Proposition 2. *Let A be a C^*-algebra and let $x \in A$ be a normal element. Then $||x^n|| = ||x||^n$ for every positive integer n.*

Proof. It will suffice to show that $||x^n||^2 = ||x||^{2n}$. Applying the C^*-identity, we can rewrite this as $||(x^n)^* x^n|| = ||x^* x||^n$. Since x is normal, the left hand side can be rewritten $||(x^* x)^n||$. We may therefore replace x by $x^* x$ and thereby reduce to the case where x is Hermitian. In this case, the C^*-identity gives $||x^2|| = ||x||^2$. Iterating this argument, we obtain

$$||x^{2^k}|| = ||x||^{2^k}.$$

Choose an integer m such that $m + n$ is a power of 2. We then have

$$||x^{m+n}|| = ||x^m x^n|| \leq ||x^m|| \, ||x^n|| \leq ||x||^m \, ||x||^n = ||x||^{m+n}.$$

Since equality holds, we must have equality throughout. Assuming $x \neq 0$, this gives

$$||x^m|| = ||x||^m \qquad ||x^n|| = ||x||^n.$$

□

Corollary 3. *Let A be a C^* algebra and let $x \in A$ be a normal element. Then the spectral radius $\rho(x)$ coincides with the norm $||x||$.*

Proof. Combine the spectral radius formula (Theorem **??**) with Proposition 2. □

For any commutative Banach algebra A, each element $x \in A$ determines a continuous map $\operatorname{Spec} A \to \mathbf{C}$, given by $\chi \mapsto \chi(x)$. This map is an algebra homomorphism, called the *Gelfand transform*.

Proposition 4. *Let A be a commutative C^*-algebra. Then the Gelfand transform $u : A \to C^0(\operatorname{Spec} A)$ is an isomorphism of C^*-algebras.*

Proof. We first show that u is a map of $*$-algebras. Equivalently, we claim that every character $\chi : A \to \mathbf{C}$ satisfies $\chi(x^*) = \overline{\chi(x)}$. It will suffice to show that χ carries Hermitian elements $x \in A$ to real numbers. Define $f : \mathbb{R} \to A$ by the formula

$$f(t) = e^{itx} = \sum_n \frac{(itx)^n}{n!}.$$

Then f satisfies $f(t)^{-1} = f(-t) = f(t)^*$, so that the C^*-identity gives

$$||f(t)||^2 = ||f(t)^* f(t)|| = 1.$$

Since χ is continuous and has norm ≤ 1, we obtain

$$1 \geq |\chi f(t)| = e^{it\chi(x)}.$$

Since this is true for both positive and negative values of t, we must have $\chi(x) \in \mathbb{R}$.

We now note that the Gelfand transform u is isometric: for $x \in A$ we have

$$||u(x)|| = \sup\{\chi \in \operatorname{Spec} A : |\chi(x)|\} = \rho(x) = ||x||$$

by Corollary 3. It follows that u is an isomorphism from A onto a closed $*$-subalgebra of $C^0(\operatorname{Spec} A)$. This subalgebra separates points: if $\chi, \chi' \in \operatorname{Spec} A$ are distinct, then we can choose $x \in A$ such that $\chi(x) \neq \chi'(x)$. Applying the Stone-Weierstrass theorem, we deduce that the image of u is the whole of $C^0(\operatorname{Spec} A)$, so that u is an isomorphism. □

Corollary 5. *Every commutative C^*-algebra is isomorphic to $C^0(X)$ for some compact Hausdorff space X. Moreover, we can canonically recover X as the spectrum* $\operatorname{Spec} A$.

Corollary 8 suggests the possibility that many familiar properties of continuous functions can be generalized to the setting of elements of an arbitrary C^*-algebra. The following provides an example:

Definition 6. Let A be a C^*-algebra and let $x \in A$. We say that x is *positive* if x is Hermitian and $\sigma(x) \subseteq \mathbb{R}_{\geq 0}$.

Example 7. Let $A = C^0(X)$ be a commutative C^*-algebra. Then a function $f \in C^0(X)$ is a positive element of A if and only if the image of f is contained in $\mathbb{R}_{\geq 0}$: that is, f is a nonnegative function.

We might try to verify properties of positive elements by restricting to the commutative case: note that if $x \in A$ is Hermitian (or, more generally, normal) then the smallest C^*-subalgebra of A containing x is commutative, hence of the form $C^*(X)$ for some compact space X. We will need to know that restriction to this subalgebra does not change the notion of positivity. In fact, we have the following more general observation:

Proposition 8. *Let A be a C^*-algebra containing a sub-C^*-algebra A_0. For each Hermitian element $x \in A_0$, the spectrum of x does not depend on whether we regard x as an element of A_0 or an element of A.*

Remark 9. The assumption that x is Hermitian is not necessary, but it will be necessary for our proof.

Proof. Replacing x by $x - \lambda$ if necessary, we are reduced to proving the following:

($*$) The element x is invertible in A if and only if it is invertible in A_0.

In other words, we must show that if x admits an inverse $x^{-1} \in A$, then that element belongs to A_0. Replace A by the C^*-subalgebra generated by x and x^{-1}, and A_0 by the C^*-subalgebra generated by x, so that A and A_0 are commutative. The inclusion $i : A_0 \hookrightarrow A$ induces a map $f : \operatorname{Spec} A \to \operatorname{Spec} A_0$. Since i is injective, the map f is surjective (otherwise, i annihilates a function supported on the open set $\operatorname{Spec} A_0 - f(\operatorname{Spec} A)$). It follows that a continuous function on $\operatorname{Spec} A_0$ is invertible if and only if its restriction to $\operatorname{Spec} A$ is invertible. Applying Proposition 4, we deduce that x is invertible in $A_0 \simeq C^0(\operatorname{Spec} A_0)$. □

Corollary 10. *Let A be a C^*-algebra and let $x \in A$ be Hermitian. Then $\sigma(x) \subseteq \mathbb{R}$.*

Proof. Using Proposition 8, we can reduce to the case where $A \simeq C^0(X)$, in which case the result is obvious. □

Corollary 11. *Let A be a C^*-algebra and let $x \in A$ be a Hermitian element. The following conditions are equivalent:*

(1) *The element $x \in A$ is positive.*

(2) *There exists a positive element $y \in A$ such that $x = y^2$.*

If these conditions are satisfied, then the element y is unique.

Proof. Suppose (2) is satisfied. Replacing A by the C^*-subalgebra generated by y, we can assume that A is commutative, hence of the form $C^0(X)$ for some compact space X. It follows immediately that x is positive.

Conversely, suppose that (1) is satisfied. Let $B \subseteq A$ be the C^*-subalgebra generated by x. Then $B \simeq C^0(Y)$ for some compact space Y and x corresponds to a nonnegative function on Y. It follows that we can write $x = y^2$ for a unique positive element $y \in B$. Suppose $y' \in A$ is any other positive element satisfying $y'^2 = x$; we wish to prove $y = y'$. Replacing A by the C^*-subalgebra generated by y' (which contains $x = y'^2$, hence B, and therefore also y), we can assume that A is commutative, so that $A \simeq C^0(X)$. In this case, the uniqueness is obvious. □

We will say that an element $x \in A$ is *negative* if $-x$ is positive. Note that if x is both positive and negative, then $\sigma(x) \subseteq \mathbb{R}_{\geq 0} \cap \mathbb{R}_{\leq 0} = \{0\}$, so that $||x|| = \rho(x) = 0$ and therefore $x = 0$.

Proposition 12. *Let A be a C^*-algebra and let $x \in A$ be Hermitian. Then x can be written uniquely in the form $x_+ + x_-$, where x_+ is positive, x_- is negative, and $x_+ x_- = x_- x_+ = 0$.*

Proof. Assume first that A is commutative, so that $A \simeq C^0(Y)$ for some compact space Y. Then x can be identified with a continuous function $f : Y \to \mathbb{R}$, and we take x_+ and x_- to correspond to the functions

$$f_+(y) = \begin{cases} f(y) & \text{if } f(y) \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$f_-(y) = \begin{cases} f(y) & \text{if } f(y) \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that x_+ and x_- are the unique elements of A having the desired properties.

We now treat the general case. Let B be the C^*-subalgebra of A generated by x. Then B is commutative, so we can find a unique pair of elements $x_-, x_+ \in B$ satisfying our requirements. This proves existence. For the uniqueness, suppose we are given another decomposition $x = x'_+ + x'_-$, where x'_+ is positive, x'_- is negative, and $x'_+ x'_- = x'_- x'_+ = 0$. We wish to prove that $x'_+ = x_+$ and $x'_- = x_-$. Replacing A by the C^*-subalgebra generated by x'_+ and x'_- (which contains x, hence also B, hence x_+ and x_-) we can reduce to the commutative case handled above. □

Here is a useful criterion for positiveness:

Lemma 13. *Let A be a C^*-algebra and let $x \in A$ be Hermitian. The following conditions are equivalent:*

(1) *The element x is positive.*

(2) *For every real number $C \geq ||x||$, we have $||C - x|| \leq C$.*

(3) *There exists a real number $C \geq ||x||$ such that $||C - x|| \leq C$.*

Proof. Replacing A by the C^*-subalgebra generated by x, we can assume that A is commutative, hence $A \simeq C^0(X)$ for some compact space X. In this case, the result is easy. $\square$

Remark 14. It follows from Lemma 13 that the collection of positive elements of A form a closed subset of A.

Proposition 15. *Let A be a C^*-algebra and let $x, y \in A$ be positive elements. Then $x + y$ is positive.*

Proof. It is clear that $x + y$ is Hermitian. Choose $C_1 \geq ||x||$ and $C_2 \geq ||y||$, so that $C = C_1 + C_2 \geq ||x + y$. Then

$$||C - x - y|| = ||(C_1 - x) + (C_2 - y)|| \leq ||C_1 - x|| + ||C_2 - y|| \leq C_1 + C_2 = C,$$

so that $x + y$ is positive by Lemma 13. $\square$