Math 261y: von Neumann Algebras (Lecture 29)

November 13, 2011

Let A be a $*$-algebra (in the purely algebraic sense). Then we there exists a canonical A-A bimodule M, given by A itself. For $x \in A$, let $l_x, r_x : M \to M$ denote the operations given by left and right multiplication by x. The $*$-operator on A determines a $\mathbf{C}$-antilinear map $J : M \to M$. Moreover, this map exchanges left and right multiplication in the following sense: for $x \in A$ and $y \in M$, we have

$$J(l_x y) = J(xy) = (xy)^* = y^* x^* = r_{x^*}(y^*) = r_{x^*} J y.$$

That is, we have $r_{x^*} = J l_x J$.

This purely algebraic construction has the following features:

(1) All left A-module maps from M to itself are given by the right action of A on M, and all right A-module maps from M to itself and given by the left action of A on M.

(2) If z belongs to the center of A, then l_z and r_z coincide.

Our goal in the next few lectures is to develop Tomita-Takesaki theory, which reconstructs an analogous picture in the setting of von Neumann algebras. If A is a von Neumann algebra, we are generally interested in bimodules which are themselves Hilbert spaces. Usually, we cannot view A as a bimodule over itself in this sense. However, we can construct something which is very analogous. Suppose that V is a representation of A containing a cyclic and separating vector v. In the last lecture, we studied the unbounded operator $S_0 : V \to V$ with domain Av, given by $S_0(xv) = x^* v$. We saw that this operator is closable, and that its closure S is injective, densely defined, and has dense image. It follows that S admits a spectral decomposition $S = J\Delta^{\frac{1}{2}}$, where J is antiunitary and $\Delta^{\frac{1}{2}}$ is a self-adjoint unbounded $\mathbf{C}$-linear operator.

Definition 1. Let $F : V \to W$ be an unbounded operator between Hilbert spaces. Assume that F is injective on the domain V_0 of F. Then we can define a new unbounded operator $F^{-1} : W \to V$ with domain $F(V_0)$, given by $F^{-1}(Fv) = v$. Note that the graphs of F and F^{-1} are identical (as subsets of $V \oplus W$, so that F^{-1} is closed if and only if F is closed. If not, then the closures of F and F^{-1} agree (provided both are defined). That is, if F is closable and its closure $\overline{F}$ is injective, then F^{-1} is closable and $\overline{F}^{-1} = \overline{F^{-1}}$.

In our situation, we have $S_0^{-1} = S_0$ (since the operation $x \mapsto x^*$ is its own inverse). It follows that $S^{-1} = S$. Let $\Delta^{-\frac{1}{2}}$ denote the inverse of $\Delta^{\frac{1}{2}}$. Then we get

$$J\Delta^{\frac{1}{2}} = S = S^{-1} = \Delta^{-\frac{1}{2}} J^{-1} = J^{-1}(J\Delta^{-\frac{1}{2}} J^{-1})$$

From the uniqueness of the polar decomposition we deduce the following:

Proposition 2. *The operator J is an antiunitary involution (that is, $J^2 = \mathrm{id}$), and $J\Delta^{-\frac{1}{2}} = \Delta^{\frac{1}{2}} J$.*

Next week, we will prove the following result:

Theorem 3. *Let A be a von Neumann algebra, V a representation of A with a cyclic and separating vector v, and let $S = J\Delta^{\frac{1}{2}}$ be as before. Then:*

(1) *We have* $A' = JAJ$. *That is, conjugation by* J *induces a conjugate-linear isomorphism of* A *with its commutant* A'.

(2) *If* $z \in Z(A)$, *then* $Jz = z^* J$.

Remark 4. In the situation of Theorem 3, we can define a right action of A on V by means of a map $\rho : A^{op} \to B(V)$ given by

$$\rho(x)(v) = Jx^* Jv.$$

Assertion (1) says that ρ induces an isomorphism from A^{op} to the commutant A' of A, and assertion (2) says that this isomorphism is given by $z \mapsto z^*$ for $z \in Z(A) = A \cap A'$. In particular, the right action of A on V commutes with the left action of A on V, so that we can regard V as an A-A bimodule.

We would like to say that Theorem 3 furnishes us with a canonical A-A bimodule, analogous to the purely algebraic situation discussed above. However, the construction depends on a choice of pair (V, v), where V is a representation of A and $v \in V$ is a cyclic and separating vector. Note that to give a pair (V, v) where v is cyclic is equivalent to giving the (ultraweakly continuous) state $\phi : A \to \mathbf{C}$, given by $\phi(x) = (xv, v)$. Note that v is separating if and only if $(xv, xv) > 0$ for all nonzero $x \in A$. We can rewrite this as $\phi(x^* x) > 0$ for $x \neq 0$. That is, v is separating if and only if the state ϕ is *faithful*, in the sense that it does not vanish on nonzero positive elements of A. We can therefore think of Theorem 3 as constructing an A-A bimodule given a choice of faithful ultraweakly continuous state $\phi : A \to \mathbf{C}$.

Suppose we are given two different faithful (ultraweakly continuous) states $\phi, \psi : A \to \mathbf{C}$, from which we can construct a pair of representations V_ϕ and V_ψ with cyclic vectors v_ϕ and v_ψ. We would like to compare these representations. To this end, consider the von Neumann algebra $B : M_2(A)$ of 2-by-2 matrices with coefficients in A. We define a linear functional $\phi \oplus \psi : B \to \mathbf{C}$ by the formula

$$(\phi \oplus \psi)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \phi(a) + \psi(d).$$

The associated inner product on B is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \mapsto \phi(a'^* a) + \phi(c'^* c) + \psi(b'^* b) + \psi(d'^* d).$$

Let W denote the Hilbert space completion of B with respect to this inner product. We can think of the elements of W as the space of matrices

$$\begin{pmatrix} v & w \\ v' & w' \end{pmatrix}$$

with $v, v' \in V_\phi$ and $w, w' \in V_\psi$. From this description, we immediately see that $\phi \oplus \psi$ is a faithful state on B. We can think of the commutant B' as consisting of matrices $\begin{pmatrix} F & G \\ F' & G' \end{pmatrix}$ where $F \in A'_\phi$ belongs to the commutant of A in V_ϕ, $G' \in A'_\psi$, $F' \in \mathrm{Hom}_A(V_\psi, V_\phi)$, and $G \in \mathrm{Hom}_A(V_\phi, V_\psi)$. Let us apply Theorem 3 to the pairs (A, V_ϕ), (A, V_ψ), and (B, W). We obtain antiunitary involutions

$$J_\phi : V_\phi \to V_\phi \qquad J_\psi : V_\psi \to V_\psi \qquad J : W \to W.$$

Unwinding the definitions, we see that J is given by the formula

$$J\begin{pmatrix} v & w \\ v' & w' \end{pmatrix} = \begin{pmatrix} J_\phi v & U(v') \\ U'(w) & J_\psi v \end{pmatrix}$$

for some antiunitary isomorphisms $U : V_\phi \to V_\psi$ and $U' : V_\psi \to V_\phi$. Since $J^2 = 1$, we have $U' = U^{-1}$. We now compute

$$\begin{aligned} J\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} J\begin{pmatrix} v & w \\ v' & w' \end{pmatrix} &= J\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} J_\phi v & Uv' \\ U^{-1}w & J_\psi w' \end{pmatrix} \\ &= J\begin{pmatrix} 0 & 0 \\ J_\phi v & Uv' \end{pmatrix} \\ &= \begin{pmatrix} 0 & UJ_\phi v \\ 0 & J_\psi U v' \end{pmatrix}. \end{aligned}$$

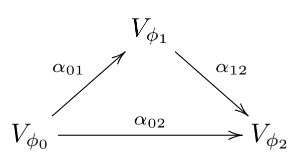
Since the operator $J\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}J$ belongs to B', it is given by some matrix $\begin{pmatrix} F & G \\ F' & G' \end{pmatrix}$ above. It follows immediately that $G = UJ_\phi = J_\psi U$. From this, we deduce that UJ_ϕ is an A-linear unitary isomorphism α from V_ϕ to V_ψ satisfying $J_\psi \circ \alpha = \alpha \circ J_\phi$. We have proven:

Proposition 5. *Let A be a von Neumann algebra, let ϕ and ψ be faithful ultraweakly continuous states on A, let V_ϕ and V_ψ be the associated representations, with antiunitary operators J_ϕ and J_ψ given in Theorem 3. Then there exists an A-linear unitary isomorphism $\alpha : V_\phi \to V_\psi$ such that $\alpha \circ J_\phi = J_\psi \circ \alpha$. In particular, α is an isomorphism of A-A bimodules.*

Remark 6. The isomorphism α is canonical. In fact, it is almost unique. Suppose we are given a pair of isomorphisms $\alpha, \beta : V_\phi \to V_\psi$ satisfying the requirements of Proposition 5. Let $\gamma = \beta^{-1} \circ \alpha$. Then γ is a unitary isomorphism of V_ϕ with itself which commutes with the action of A and with J_ϕ. It therefore commutes with the action of the commutant $A'_\phi = J_\phi A J_\phi$. It follows that $\gamma \in A \cap A'_\phi = Z(A)$. Theorem 3 then gives $J_\phi \circ \gamma = \gamma^* \circ J_\phi$, so that $\gamma = \gamma^*$. Since γ is unitary, we deduce that $\gamma^2 = 1$. If A is a factor, this means that $\gamma = \pm 1$: that is, the isomorphisms α and β differ by at most a sign.

Definition 7. Let A be a von Neumann algebra which admits an ultraweakly continuous faithful state ϕ (for example, any separable von Neumann algebra). We let $L^2(A)$ denote the Hilbert space V_ϕ, and $J : L^2(A) \to L^2(A)$ the antiunitary isomorphism appearing in Theorem 3. It follows from the above considerations that the pair $(L^2(A), J)$ is independent of the choice of ϕ up to isomorphism.

Remark 8. Since the isomorphism α appearing in Proposition 5 is not quite unique, one might worry that $L^2(A)$ is not quite well-defined. However, although α is not unique it is nevertheless *canonical*. That is, given a triple of ultraweakly continuous faithful states $\phi_0, \phi_1, \phi_2 : A \to \mathbf{C}$, the diagram of Hilbert spaces and unitary isomorphisms



is actually commutative. One can prove this by applying Theorem 3 to the state $\phi_0 \oplus \phi_1 \oplus \phi_2$ on $M_3(A)$.