Math 261y: von Neumann Algebras (Lecture 28)

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Let A be a von Neumann algebra which admits a faithful (ultraweakly continuous) finite trace ϕ. In the last lecture, we saw that the associated representation V_ϕ is equipped with an antiunitary involution J such that conjugation by J carries A isomorphically to its commutant A'. Our goal in this lecture (and those which follow) is to see how much of this picture we can reproduce without the assumption that A is finite.

Suppose we are given a realization $A \hookrightarrow B(V)$ with commutant A'. We might ask whether or not A' is even abstractly isomorphic to A^{op}. The answer is in general no: for example, if $A = B(V)$, then $A' \simeq \mathbf{C}$ is generally much smaller than A. In some sense, this is because the defining representation of $B(V)$ is too small. Let us now try to articulate the problem more precisely.

Definition 1. Let A be a von Neumann algebra, V a representation of A, and $v \in V$ a vector. Recall that v is said to be a *cyclic vector* if Av is dense in V. We say that v is a *separating vector* if it is not annihilated by any elements of A: that is, if the map $x \mapsto xv$ is an isomorphism of vector spaces $A \to Av$.

Remark 2. Let $\phi : A \to \mathbf{C}$ be an (ultraweakly continuous) state, V_ϕ the associated representation, and $v \in V_\phi$ the generating vector. For each $x \in A$, we have $(xv, xv) = \phi(x^* x)$. Thus v is a separating vector if and only if $\phi(x^* x) = 0$ implies $x = 0$. In other words, v is separating if and only if ϕ is faithful (that is, ϕ does not annihilate any nonzero positive elements of A). When ϕ is a trace, this is equivalent to the condition that the representation V_ϕ is faithful. In general, it is much stronger. For example, the tautological representation of $B(V)$ on V does not have any separating vectors unless $\dim(V) \leq 1$.

Proposition 3. *Let $A \subseteq B(V)$ be a von Neumann algebra and $v \in V$ a vector. The following conditions are equivalent:*

(1) *v is a separating vector for A.*

(2) *v is a cyclic vector for A'.*

Proof. Suppose first that (1) is satisfied. If v is not a cyclic vector for the action of A', then we can decompose V as a direct sum $\overline{A'v} \oplus W$ of A'-representations. Let $e : V \to V$ denote the orthogonal projection onto W. Then $e \in A'' = A$ and $ev = 0$, contradicting the assumption that v is separating.

Conversely, suppose that v is a cyclic vector for A'. Let $I = \{x \in A : xv = 0\}$. Then I is an ultraweakly closed left ideal of A, hence of the form Ae for some projection $e \in A$. Then V decomposes as a direct sum $eV \oplus (1-e)V$ as representations of A'. Since $eV = 0$, we have $v \in (1-e)V$. Using (2) we deduce that $(1-e)V = V$. It follows that $e = 0$, so $I = 0$ and v is a separating vector. $\square$

Example 4. Let $A = B(V)$. Then A is Morita equivalent to $\mathbf{C}$: all representations of A have the form $V \otimes W$, for some Hilbert space W. Every vector $v \in V \otimes W$ determines a bounded operator $\lambda : V \to W$, and v is separating if and only if λ is injective. In particular, a separating vector exists only when the dimension of W is at least as large as the dimension of V. If V is separable, this condition is also sufficient. But if V is not separable, then one can never find a separating vector: every $v \in V \otimes W$ belongs to $V \otimes W_0$ for some separable subspace $W_0 \subseteq W$, and the induced map $V \to W_0$ cannot be injective.

Example 4 shows that one cannot expect to find separating vectors in complete generality. However, they always exist under some mild assumptions.

Proposition 5. *Let A be a separable von Neumann algebra (that is, the predual of A is separable as a Banach space). Then there exists a representation V of A containing a cyclic and separating vector.*

Proof. It suffices to find a representation V of A containing a separating vector v (we can then replace V by $\overline{Av}$ to ensure that v is also cyclic). Choose an embedding $A \subseteq B(V_0)$, where V_0 is a separable Hilbert space. We may assume without loss of generality that V_0 has a countable orthonormal basis $e_1, e_2, \ldots$. Let W be another Hilbert space with countable orthonormal basis $f_1, f_2, \ldots$. An easy calculation shows that the vector

$$\sum \frac{e_n \otimes f_n}{n}$$

is a separating vector for the action of $B(V_0)$ on $V_0 \otimes W$, hence a separating vector for the action of A on $V_0 \otimes W$. $\square$

Remark 6. We will eventually see that if V is a representation of A containing a cyclic and separating vector, then V is unique up to (isometric) isomorphism.

Let us now suppose that V is a representation of A containing a cyclic separating vector v. Motivated by the constructions of the previous lecture, we can try to define an operator $S_0 : V \to V$ by the formula $S_0(xv) = x^*v$. This formula makes sense so long as v is separating (since x is determined by xv). However, there is no reason in general why it should be a bounded operator. We therefore need a brief digression about the theory of unbounded operators.

Definition 7. Let V and W be Hilbert spaces (over $\mathbb{R}$ or $\mathbf{C}$). An *unbounded operator* from V to W is a linear subspace $V_0 \subseteq V$ (not necessarily closed) and a linear map $F : V_0 \to W$ (not necessarily continuous). We refer to V_0 as the *domain* of F. We say that F is *densely defined* if V_0 is dense in V.

The *graph* of F is the subset $\Gamma(F) = \{(v, F(v)) : v \in V_0\} \subseteq V_0 \times W$. We say that F is *closed* if $\Gamma(F)$ is a closed subset of $V \times W$. We say that F is *closable* if the closure of $\Gamma(F)$ is the graph of an unbounded operator: that is, if the intersection $\overline{\Gamma(F)} \cap (V \times \{0\})$ is trivial. In this case, $\overline{\Gamma(F)}$ is the graph of another unbounded operator $\overline{F} : V_1 \to W$, where V_1 is a subspace of V containing V_0.

Note that if F is closed and the domain of F is equal to V, then F is automatically a bounded operator: this follows from the closed graph theorem.

Definition 8. Let F be a densely defined unbounded operator from V to W with domain V_0. We define an unbounded operator F^* from W to V as follows. The domain of F^* is the collection of those vectors $w \in W$ such that the functional $v \to (F(v), w)$ is bounded (a priori, this functional is defined only on V_0). In this case, we let $F^*(w)$ denote the unique element in v such that $(v, F^*(w)) = (F(v), w)$ for all $v \in V$. Note that F^* is automatically a closed operator. We say that F is *self-adjoint* if $F = F^*$ (in which case F is automatically closed).

We will need a few facts about polar decompositions of unbounded operators. First, let us consider the case of bounded operators.

Proposition 9. *Let $F : V \to W$ be a bounded map of Hilbert spaces which is injective with dense image. Then F factors as a composition*

$$V \stackrel{|F|}{\to} V \stackrel{U}{\to} W,$$

where $|F|$ is a positive self-adjoint operator and U is an isometric isomorphism. Moreover, this factorization is unique.

Proof. Let $|F|$ be the unique positive square root of F^*F. Since F has dense image, F^* is injective; it follows that F^*F is injective, so that $|F|$ is injective. Since $|F|$ is self-adjoint, it has dense image. For $v, v' \in V$, we have

$$(|F|v, |F|v')_V = (|F|^2 v, v')_V = (F^*Fv, v')_V = (Fv, Fv')_W,$$

Thus there is a unique isometry $U : V \to W$ such that $U(|F|v) = Fv$. Since F has dense image, U has dense image; it follows that U is an isomorphism. $\square$

Now suppose that F is a closed unbounded operator from a Hilbert space V to a Hilbert space W, and let Γ denote the graph of F. We regard Γ as a Hilbert space in its own right, and let $p : \Gamma \to V$ and $q : \Gamma \to W$ denote the projection maps. Then p is injective, and has dense image since F is densely defined. Using Proposition 9, we can identify Γ with V, so that under this identification p corresponds to a positive self-adjoint operator. If F is injective with dense image, then Proposition 9 implies that q factors as a composition

$$V \stackrel{|q|}{\to} V \stackrel{U}{\to} W.$$

Let $|F|$ denote the closed operator from V to itself given by composing F with U^{-1}. That is, the domain of $|F|$ is the image of p, and $|F|$ is given by the formula $|F|p(v) = |q|(v)$.

Since the embedding of Γ into $V \oplus W$ is an isometry, we have

$$(v, v) = (pv, pv) + (qv, qv) = (p^2 v, v) + (|q|^2 v, v)$$

for all $v \in V$. It follows that $p^2 = 1 - |q|^2$, so that both p and $|q|$ belong to the commutative von Neumann subalgebra of $B(V)$ generated by p^2. It follows that p and $|q|$ commute with each other.

We claim that the operator $|F|$ is self-adjoint. To prove this, we note that $v' = |F|^* v$ if and only if, for every $v'' \in V$, we have $(|q|(v''), v) = (pv'', v')$. Since p and $|q|$ are self adjoint, we can rewrite this equation as $(v'', |q|v) = (v'', pv')$. This is satisfied for all $v'' \in V$ if and only if $pv' = |q|v$. In this case, set $u = pv + |q|v'$. Then

$$pu = p^2 v + p|q|v' = p^2 v + |q|pv' = p^2 v + |q|^2 v = v$$

$$|q|u = |q|pv + |q|^2 v' = p|q|v + |q|^2 v' = p^2 v' + |q|^2 v' = v'$$

from which it follows that $|F|(v) = v'$. Conversely, if $|F|(v) = v'$, then there exists $u \in V$ such that $v = pu$ and $v' = |q|u$, from which the equality $p|q| = |q|p$ gives $pv' = |q|v$. This proves:

Proposition 10 (Polar Decomposition, Unbounded Version)**.** *Let $F : V \to W$ be a closed unbounded operator between Hilbert spaces. Assume that F is densely defined, injective, and has dense image. Then F factors as a composition*

$$V \stackrel{|F|}{\to} V \stackrel{U}{\to} W$$

where U is a unitary isomorphism and $|F|$ is a positive self-adjoint unbounded operator (this means that $(|F|v, v) \geq 0$ for all v in the domain of $|F|$: in our case, it follows from the calculation

$$(|F|pv, pv) = (|q|v, pv) = (p|q|v, v) \geq 0,$$

since $p|q|$ is a product of commuting positive operators and therefore positive).

Remark 11. It is not hard to see that the factorization $F = U|F|$ of Proposition 10 is unique. It is called the *polar decomposition* of F.

Remark 12. Proposition 10 is also valid for real Hilbert spaces (this can be deduced from the complex case, by passing to the complexification).

Let us now return to the case of interest: A is a von Neumann algebra acting on a representation V with a cyclic separating vector v. We define an unbounded operator $S_0 : V \to V$ with domain Av by the formula

$$S_0(xv) = x^* v.$$

Proposition 13. *In the situation above, the operator S_0 is closable. Moreover, the closure S of S_0 is injective.*

Proof. Let Γ_0 denote the graph of S_0 and Γ its closure. For each $f \in A'$ and $x \in A$, we have

$$(S_0(xv), fv) = (x^* v, fv) = (v, xfv) = (v, fxv) = (f^* v, xv).$$

It follows by continuity that for $(w, w') \in \Gamma$ we have $(w', fv) = (f^* v, w)$. In particular, if $w = 0$, then $(w', fv) = 0$ for all $f \in A'$. Since v is a separating vector for the action of A on V, it is a cyclic vector for the action of A' on V: that is, $w' = 0$. This proves that $\overline{\Gamma}$ is the graph of an unbounded operator: that is, S_0 is closable. The same argument shows that if $w' = 0$, then $w = 0$: that is, the closure S of S_0 is injective. $\square$

Applying Proposition 10 to the closed unbounded operator $S : V \to V$ (as a map of real Hilbert spaces) we deduce that S admits a factorization $S = J\Delta^{\frac{1}{2}}$, where J is an isometry (of real Hilbert spaces) and $\Delta^{\frac{1}{2}}$ is a self-adjoint unbounded operator. Since S_0 is $\mathbf{C}$-antilinear, S has the same property. If we regard $i \in \mathbf{C}$ as a real linear operator from V to itself, we get

$$(iJ)\Delta^{\frac{1}{2}} = iS = -Si = -J\Delta^{\frac{1}{2}} i = (-Ji)(i^{-1}\Delta^{\frac{1}{2}} i).$$

From the uniqueness of the polar decomposition, we deduce

$$i^{-1}\Delta^{\frac{1}{2}} i = \Delta^{\frac{1}{2}} \qquad iJ = -Ji.$$

That is, $\Delta^{\frac{1}{2}}$ is a $\mathbf{C}$-linear operator, and J is $\mathbf{C}$-antilinear. We will study the situation further in the next lecture.