Math 261y: von Neumann Algebras (Lecture 27)

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Let A be a von Neumann algebra equipped with an ultraweakly continuous state $\phi : A \to \mathbf{C}$. We can use ϕ to construct a Hilbert space V_ϕ, given by completing A with respect to the inner product

$$(x, y) = \phi(y^* x).$$

Then V_ϕ is a representation of A, equipped with a cyclic vector which we will denote by v (so that the map from A into V_ϕ is given by $x \mapsto xv$).

Question 1. In terms of the above construction, what does it mean for the state ϕ to be tracial?

Definition 2. Let V be a Hilbert space. We say that a map $J : V \to V$ is *antiunitary* if J is $\mathbf{C}$-antilinear and satisfies

$$(v, w) = \overline{(Jv, Jw)} = (Jw, Jv)$$

for each $v, w \in V$. We say that J is an *antiunitary involution* if J^2 is the identity map. In this case, we can rewrite the above identity as

$$(Jv, w) = (Jw, v).$$

Proposition 3. (1) *Let ϕ be an (ultraweakly continuous) finite trace on a von Neumann algebra A. Then the construction $xv \mapsto x^*v$ extends continuously to an antiunitary involution on the Hilbert space V_ϕ.*

(2) *Let A be a von Neumann algebra and let $V \in \mathrm{Rep}(A)$. Let $v \in V$ be a vector, and suppose that there exists an antiunitary map $J : V \to V$ such that $J(xv) = x^*v$) for all $x \in A$. Then the map $\phi : A \to \mathbf{C}$ given by $\phi(x) = (xv, v)$ is a finite trace on A.*

Proof. Suppose first that ϕ is a finite trace, and let $v \in V_\phi$ be the canonical vector. We then have

$$(xv, yv) = \phi(y^*x) = \phi(xy^*) = (y^*v, x^*v) = \overline{(x^*v, y^*v)},$$

so that the map $xv \mapsto x^*v$ extends to an antiunitary map from V_ϕ to itself. This map is obviously an involution.

Now suppose that V is an arbitrary representation of A equipped with a vector $v \in V$ and an antiunitary map J satisfying the requirements of (2). We then have

$$\phi(xy) = (xyv, v) = (yv, x^*v) = (Jx^*v, Jyv) = (xv, y^*v) = (yxv, v) = \phi(yx)$$

so that ϕ is a trace. □

Remark 4. In the situation of part (2) of Proposition 3, suppose in addition that $v \in V$ is a cyclic vector. Then V is canonically isomorphic to the Hilbert space V_ϕ constructed from the state ϕ. Since $J^2xv = J(x^*v) = xv$ for all $x \in A$, we conclude that J^2 is the identity on V: that is, J is automatically involutive.

Let ϕ be a finite ultraweakly continuous trace on A, and regard V_ϕ as a representation of A. For each $x \in A$, let l_x denote the operator on V_ϕ induced by the left action of A on itself. That is, we have

$$l_x(yv) = (xy)v.$$

We then compute

$$(Jl_xJ)(yv) = Jl_x(y^*v) = J((xy^*)v) = (yx^*)v = r_{x^*}(yv),$$

where $r_{x^*} : V_\phi \to V_\phi$ is given by continuously extending the map given by right multiplication by x^*. If we regard l as a map of $*$-algebras $A \to B(V_\phi)$, then then conjugation by J carries l to another map of $*$-algebras $A^{op} \to B(V_\phi)$. Let A' denote the commutant of A in $B(V_\phi)$. Since the operators l_x and r_y commute for $x, y \in A$, we can regard r as a map of von Neumann algebras $A^{op} \to A'$.

Remark 5. More generally, let A be any von Neumann algebra equipped with a representation $\rho : A \to B(V)$, a cyclic vector $v \in V$, and an antiunitary map $J : V \to V$ satisfying $J(xv) = x^*v$ for $x \in A$. Then $V \simeq V_\phi$ where ϕ is the trace given by $\phi(x) = (xv, v)$. It follows that $x \mapsto J\rho(x)J$ determines a map $\rho' : A^{op} \to B(V)$, whose image is contained in the commutant A' of the image of the original map ρ.

We now prove a result which was promised several lectures back:

Proposition 6. *Let ϕ be an ultraweakly continuous finite trace on a von Neumann algebra A, and let V_ϕ denote the corresponding representation. Then the above construction induces a surjection $\rho : A^{op} \to A'$. In particular, if ϕ is faithful, we get an isomorphism $A^{op} \simeq A'$.*

Proof. Replacing A by a direct factor if necessary, we may assume without loss of generality that ϕ is faithful. We will identify A with its image in $B(V_\phi)$ (under the representation given by the left action of A on itself). Let us regard V_ϕ as a representation of A'. For each $x \in A$, the operator r_x (given by $yv \mapsto yxv$) belongs to A'. In particular, $xv = r_x(v) \in A'v$, so that $Av \subseteq A'v$. It follows that v is a cyclic vector for V_ϕ, regarded as a representation of A'.

Let $J : V_\phi \to V_\phi$ be the antiunitary involution constructed above (given by $xv \mapsto x^*v$ for $x \in A$). We claim that J satisfies the hypothesis of part (2) of Proposition 3, with respect to the action of A' on V_ϕ. That is, we claim that if $F \in A'$, then we have $JF(v) = F^*v$. Since V_ϕ is topologically generated by elements of the form xv, it will suffice to show that

$$(JF(v), xv) = (F^*(v), xv)$$

for each $x \in A$. We now compute

$$\begin{aligned}(JF(v), xv) &= (Jxv, F(v)) \\ &= (x^*v, F(v)) \\ &= (v, xF(v)) \\ &= (v, F(xv)) \\ &= (F^*(v), xv).\end{aligned}$$

Applying Remark 5, we deduce that for each $F \in A'$, we have $JFJ \in A'' = A$. It follows that $F = JxJ$ for some $x \in A$: that is, F belongs to the image of the map $A^{op} \to A'$. □

Remark 7. Let A be a von Neumann algebra acting on a Hilbert space V, let $v \in V$ be a cyclic vector, and let $J : V \to V$ be an antiunitary map. The equation $J(xv) = x^*v$ is equivalent to the assertion that, for each $w \in V$, we have

$$(Jxv, w) = (x^*v, w).$$

We can rewrite the left hand side as (Jw, xv) and the right hand side as (v, xw). Consequently, the hypothesis of (2) of Proposition 3 can be written as

$$(Jw, xv) = (v, xw)$$

for all $x \in A$. Both sides of this equation are ultraweakly continuous functions of x. Consequently, if $A_0 \subseteq A$ is a $*$-algebra generating A, then it suffices to verify the equation $J(xv) = x^* v$ for elements $x \in A_0$.

Definition 8. Let G be a locally compact group, and choose a Haar measure μ on G. We let $L^2(G) = L^2(G, \mu)$ denote the space of square-integrable functions on G. Note that G acts on $L^2(G)$ by left translation. The *group von Neumann algebra of* G is the von Neumann algebra in $B(L^2(G))$ generated by the image of G. We will denote this von Neumann algebra by $A(G)$.

In other words, the group von Neumann algebra $A(G)$ is the ultraweak closure of the image of the group ring $\mathbf{C}[G]$ inside $B(L^2(G))$ (we regard $\mathbf{C}[G]$ as a $*$-algebra, with involution given by $(\sum \lambda_g g)^* = \sum \overline{\lambda}_g g^{-1}$).

Let us now suppose that the group G is discrete. Let $e \in G$ be the identity element. For each $g \in G$, let $\chi_g \in L^2(G)$ denote the characteristic function of the set $\{g\}$, so that

$$\chi_g(h) = \begin{cases} 1 & \text{if } g = h \\ 0 & \text{if } g \neq h. \end{cases}$$

Let $l_g, r_g : L^2(G) \to L^2(G)$ be the maps given by left and right translation, so that

$$l_g(\chi_h) = \chi_{gh} \qquad r_g(\chi_h) = \chi_{hg}.$$

We let $J : L^2(G) \to L^2(G)$ denote the antiunitary map given by $(Jf)(g) = \overline{f(g^{-1})}$ (so that $J(\chi_g) = \chi_{g^{-1}}$). Note that

$$J(l_g \chi_e) = J(\chi_g) = \chi_{g^{-1}} = l_{g^{-1}} \chi_e = l_g^* \chi_e.$$

Since the elements l_g generate $A(G)$ as a von Neumann algebra, it follows from Remark 7 that the triple $(L^2(G), J, \chi_e)$ satisfies the hypothesis of part (2) of Proposition 3. In particular, we obtain a finite trace

$$\phi : A(G) \to \mathbf{C},$$

given by $\phi(x) = (x\chi_e, \chi_e)$. We can characterize ϕ as the unique ultraweakly continuous function whose restriction to the group algebra $\mathbf{C}[G]$ is given by

$$\phi(\sum \lambda_g g) = \lambda_e.$$

Applying Proposition 6, we see that conjugation by J induces an antiisomorphism from $A(G)$ to the commutant $A(G)'$. We have

$$(J l_g J)(\chi_h) = J l_g(\chi_{h^{-1}}) = J \chi_{gh^{-1}} = \chi_{hg^{-1}} = r_{g^{-1}} \chi_h.$$

It follows by continuity that $J l_g J = r_{g^{-1}}$. Thus $A(G)' = JA(G)J$ is the von Neumann algebra of operators on $L^2(G)$ given by right translation by the elements of G.

Since $A(G)$ acts faithfully on $L^2(G)$, the trace $\phi : A(G) \to \mathbf{C}$ is faithful. It follows that the action of $A(G)$ on the element $\chi_e \in L^2(G)$ is faithful. In particular, we get an injective map $A(G) \to L^2(G)$, given by

$$x \mapsto x(\chi_e).$$

In particular, we can identify elements of $A(G)$ with possibly infinite formal sums

$$\sum \lambda_g g,$$

where the coefficients λ_g are square-summable (not every such sequence needs to come from an element of $A(G)$, however).

Proposition 9. *Let G be a nontrivial discrete group in which every non-identity conjugacy class is infinite. Then $A(G)$ is a factor of type II_1.*

Proof. Let $x = \sum \lambda_g g$ be an element of $A(G)$. If x is central, then it commutes with every element of G. This implies that the function $g \mapsto \lambda_g$ is invariant under conjugation. Since every nontrivial conjugacy class of G is infinite, the square summability of the coefficients λ_g implies that $\lambda_g = 0$ for $g \neq e$. Thus the center of $A(G)$ is one-dimensional, so that $A(G)$ is a factor.

We have seen above that the von Neumann algebra $A(G)$ admits a finite trace, and is therefore finite. Consequently, $A(G)$ is either of type I or type II. If it is of type I, then it is isomorphic to $B(V)$ for some Hilbert space V. The finiteness of $A(G)$ then implies that V is finite-dimensional, so that $A(G) \simeq B(V)$ is finite dimensional. But G contains an infinite conjugacy class, so that the group algebra $\mathbf{C}[G]$ is infinite dimensional. Since this group algebra injects into $A(G)$, we obtain a contradiction. It follows that $A(G)$ is a factor of type II. Since it is finite, it is of type II_1. □

Corollary 10. *There exist factors of type II.*

Proof. It is easy to find examples of groups G satisfying the requirements of Proposition 9. For example, we can take G to be a free group on two generators. □