Math 261y: von Neumann Algebras (Lecture 26)

November 1, 2011

Our goal in this lecture is to prove the following result, which was asserted without proof in the last lecture:

Theorem 1 (Ryll-Nardzewski)**.** *Let M be a Banach space, let K be a convex subset of M which is compact with respect to the weak topology on M, and let G be a group of isometries of M which preserves K. Then there is an element of K which is fixed by the action of G.*

As a warm-up, we prove the following simpler result:

Proposition 2. *Let M be a Banach space, let K be a convex subset of M which is compact with respect to the weak topology on M, and let $F : M \to M$ be a bounded linear map which preserves K (not necessarily an isometry). Then there is an element $x \in K$ satisfying $F(x) = x$.*

Proof. For each integer n, let F_n denote the bounded linear map

$$x \mapsto \frac{1}{n}(x + F(x) + F^2(x) + \cdots + F^{n-1}(x)).$$

Since K is convex, each of these maps carries K into itself. Let $K_n = F_n(K) \subseteq K$, so that K_n is a weakly compact subset of K. We first claim that the intersection $\bigcap K_n$ is nonempty. Since K is weakly compact, it will suffice to show that each finite intersection

$$K_{n_1} \cap \cdots \cap K_{n_m}$$

is nonempty. This follows from the observation that this intersection contains

$$F_{n_1}(F_{n_2}(\cdots(F_{n_m}(x))))$$

for each $x \in K$ (note that the operators F_j commute with one another).

Let $x \in \bigcap K_n$. For each integer y, we can write $x = F_n(y)$ for some $y \in K$. It follows that

$$F(x) - x = F(\frac{y + \cdots + F^{n-1}y}{n}) - \frac{y + \cdots + F^{n-1}(y)}{n} = \frac{1}{n}(F^n(y) - y) \in \frac{1}{n}(K - K).$$

Since $K - K$ is weakly compact, every weakly open neighborhood of 0 in M contains $\frac{1}{n}(K - K)$ for n large enough. It follows that $F(x) - x$ belongs to every weakly open neighborhood of the origin, so that $F(x) = x$. □

We now turn to the proof of Theorem 1. The first observation is:

(a) We may assume without loss of generality that G is finitely generated.

Write G as a union of finitely generated subgroups G_α. Then the fixed point set K^G is given by the intersection $\bigcap_\alpha K^{G_\alpha}$. By compactness, if each K^{G_α} is nonempty, then K^G will be nonempty.

Let us suppose that G contains elements $g_1, \ldots, g_n \in G$. Let $F : M \to M$ be the linear map given by $F(x) = \frac{g_1(x)+\cdots+g_n(x)}{n}$, so that F carries K into itself. Using Proposition 2, we can choose an element $x \in K$ such that $F(x) = x$. We will prove that $g_i(x) = x$ for all x. Taking the g_i to be a set of generators for the group G, we will obtain a proof Theorem 1.

Suppose otherwise. We may assume without loss of generality that there exists some integer $1 \leq m \leq n$ such that $g_i(x) \neq x$ for $i \leq m$, and $g_i(x) = x$ for $i > m$. Then

$$x = F(x) = \frac{1}{n}(\sum_{1 \leq i \leq m} g_i(x)) + \frac{n-m}{n}x$$

so that

$$\frac{m}{n}x = \frac{1}{n}\sum_{1 \leq i \leq m} g_i(x)$$

and therefore x is fixed by the operator $y \mapsto \frac{g_1(y)+\cdots+g_m(y)}{m}$. We may therefore replace the sequence $\{g_1, \ldots, g_n\}$ by $\{g_1, \ldots, g_m\}$, and thereby reduce to the case where $g_i(x) \neq x$ for all i.

To obtain a contradiction, we are free to replace G by the group generated by the elements $g_1, \ldots, g_m$, and K by the closed convex hull of the orbit $Gx \subseteq K$ (in the weak topology). In particular, K is contained in the closed subspace of M generated by a countable set of vectors. Replacing M by this closed subspace, we may assume that M is separable.

Choose a real number $\epsilon > 0$ such that $||g_i(x) - x|| > \epsilon$ for each i. We will need the following technical lemma:

Lemma 3. *There exists a weakly compact convex subset $K' \subsetneq K$ such that the difference $K - K'$ has diameter $\leq \epsilon$.*

Let us assume Lemma 3 for the moment. Since K is the closed convex hull of Gx and $K' \subsetneq K$ is closed and convex, there must exist an element $h \in G$ such that $hx \notin K'$. Then

$$hx = hF(x) = \frac{hg_1(x) + \cdots + hg_n(x)}{n} \notin K'.$$

Since K' is convex, this implies that $hg_i(x) \notin K'$ for some i. Then $hx, hg_i(x) \in K - K'$. Since $K - K'$ has diameter $\leq \epsilon$, we conclude that $||hg_i(x) - h(x)|| \leq \epsilon$. Since h is an isometry, we obtain $||g_i(x) - x|| \leq \epsilon$, contradicting our assumption.

It remains to prove Lemma 3. Let E denote the set of extreme points of K (that is, points which do not lie on the interior of any line segment contained in K). Since K is compact (in the weak topology), the Krein-Milman theorem asserts that K is the closed convex hull of E. Let $\overline{E} \subseteq K$ denote the weak closure of E. Let B denote the closed ball of radius $\frac{\epsilon}{3}$ around the origin. Note that B is also closed in the weak topology (since $y \in B$ if and only of $|\phi(y)| \leq \frac{\epsilon}{3}$ for all linear functionals ϕ of norm 1). Since M is separable, there exists a countable collection of points $y_i \in M$ such that the sets $y_i + B$ cover M. In particular, the intersections

$$(y_i + B) \cap \overline{E}$$

give a countable covering of $\overline{E}$ by weakly closed subsets. Since $\overline{E}$ is weakly compact, the Baire category theorem implies that one of the sets $(y_i + B) \cap \overline{E}$ has nonempty interior U in $\overline{E}$ (with respect to the weak topology),

Let K_1 be the closed convex hull of $\overline{E} - U$ and let K_2 be the closed convex hull of $(y_i + B) \cap \overline{E}$. Then K_1 and K_2 are closed convex subsets of K. Since K is the closed convex hull of $E \subseteq (\overline{E} - U) \cup (y_i + B)$, it is the convex join of K_1 and K_2. That is, K can be described as the image of the map

$$K_1 \times K_2 \times [0, 1] \to M$$

$$(v, w, t) \mapsto tv + (1-t)w.$$

For $\delta > 0$, let $K(\delta)$ denote the image of the restriction of this map to $K_1 \times K_2 \times [\delta, 1]$. We claim that if δ is small enough, then $K(\delta)$ has the desired properties. It is clear that each K_δ is a weakly closed convex subset of K. We are therefore reduced to proving two things:

(i) For δ sufficiently small, the set $K - K(\delta)$ has diameter $\leq \epsilon$. Note that K is contained in a ball of some finite radius C (when regarded as a set of linear operators on $M^\vee$), K is pointwise bounded by compactness, hence uniformly bounded). If $y, y' \in K - K_\delta$, then we can write

$$y = tv + (1-t)w \qquad y' = t'v' + (1-t')w'$$

for $t, t' < \delta$. Then

$$||y - y'|| \leq t||v|| + t||w|| + t'||v'|| + t'||w'|| + ||w - w'|| \leq 4tC + \frac{2}{3}\epsilon \leq 4\delta C + \frac{2}{3}\epsilon,$$

where the bound on $||w - w'||$ comes from the observation that $K_2 \subseteq y_i + B$ has diameter $\frac{2}{3}\epsilon$. Choosing $\delta < \frac{\epsilon}{12C}$ will achieve the desired result.

(ii) The set $K(\delta)$ is distinct from K if δ is positive. Since U is a nonempty open subset of $\overline{E}$, it contains some element $y \in E$. We claim that $y \notin K(\delta)$: that is, we cannot write $y = tv + (1-t)w$ where $t \leq 1 - \delta$, $v \in K_1$, and $w \in K_2$. Since y is an extreme point of K, it will suffice to show that $y \notin K_1$.

Since the weak topology on M is locally convex, we can choose a (weakly) open convex set $V \subseteq M$ whose (weak) closure $\overline{V}$ satisfies $(y - \overline{V}) \cap \overline{E} \subseteq U$. Since $\overline{E} - U$ is compact, it admits a finite covering by weakly open sets $z_1 + V, z_2 + V, \cdots, z_k + V$ where $z_i \in \overline{E}$. It follows that K_1 is contained in the closed convex hull of $\bigcup((z_i + V) \cap \overline{E})$, which is contained in the convex join of the sets $(z_i + \overline{V}) \cap K$. If $y \in K_1$, then since y is an extreme point of K, we deduce that $y \in z_i + \overline{V}$ for some i. Then $z_i \in (y - \overline{V}) \cap \overline{E} \subseteq U$, contradicting our assumption that $z_i \in \overline{E} - U$.