Math 261y: von Neumann Algebras (Lecture 25)

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Our goal in this lecture is to begin the proof of the following result:

Theorem 1. *Let A be a finite von Neumann algebra. Then A is a product of von Neumann algebras A_α, each of which admits an ultraweakly continuous, faithful finite trace.*

The proof proceeds in several steps.

Lemma 2. *Let A be a von Neumann algebra and let $\phi : A \to \mathbf{C}$ be an ultraweakly continuous faithful finite trace. Then A factors as a product $A' \times A''$, where ϕ is a faithful finite trace on A' and vanishes on A''.*

Proof. Let V_ϕ denote the cyclic representation of A determined by ϕ and $v \in V_\phi$ its cyclic vector, so that $(xv, yv) = \phi(y^* x)$. Let $I \subseteq A$ be the subset consisting of elements $x \in A$ such that $xv = 0$. Then I is an ultraweakly closed left ideal of A. Note that $I = \{x \in A : \phi(x^* x) = 0\}$. Since ϕ is a trace, we have $\phi(x^* x) = \phi(x x^*)$. It follows that I is a $*$-ideal of A. It follows that $I = eA$ for some central projection $e \in A$. We claim that the decomposition $A = (1-e)A \times eA$ has the desired property. It is clear that ϕ vanishes on eA (since $eV = 0$). To see that ϕ is faithful on $(1-e)A$, suppose that $h \in (1-e)A$ is a positive element with $\phi(h) = 0$. Writing $h = x^* x$ for $x \in (1-e)A$, we obtain

$$\phi(x^* x) = (xv, xv) = 0,$$

so that $x \in I = eA$ and therefore $x = 0$. □

Let A be an arbitrary von Neumann algebra. Choose a maximal collection of mutually orthogonal central projections $e_\alpha \in A$ such that each $e_\alpha A$ admits an ultraweakly continuous faithful finite trace. Then $A = A' \times \prod_\alpha e_\alpha A$. By maximality, the von Neumann algebra A' does not admit any ultraweakly continuous finite traces. Consequently, Theorem 1 is a consequence of the following:

Proposition 3. *Let A be a nonzero finite von Neumann algebra. Then A admits an ultraweakly continuous faithful finite trace.*

To prove this, let $S(A)$ denote the collection of all ultraweakly continuous states on A, and let $U(A)$ denote the unitary group of A. Then $U(A)$ acts on $S(A)$, via the formula $\phi^u(x) = \phi(u^{-1} x u)$.

Lemma 4. *Let A be a von Neumann algebra and let $\phi : A \to \mathbf{C}$ be a state. The following conditions are equivalent:*

(1) *The state ϕ is a trace.*

(2) *The state ϕ is a fixed point for the action of $U(A)$. That is, we have $\phi^u = \phi$ for each unitary element $u \in A$.*

Proof. If (1) is satisfied, then we have

$$\phi^u(x) = \phi(u^{-1} x u) = \phi(u u^{-1} x) = \phi(x)$$

so that condition (2) is also satisfied. Conversely, suppose that ϕ satisfies (2). Then $\phi(x) = \phi(u^{-1}xu)$ for all $u \in U(A)$, $x \in A$. Taking $x = uy$, we obtain $\phi(uy) = \phi(yu)$ for each $y \in A$ and each $u \in U(A)$. We wish to prove that the analogous assertion holds for an arbitrary element $u \in A$. To prove this, it suffices to observe that A is spanned by $U(A)$ as a complex vector space. That is, every element of $z \in A$ can be written as a complex linear combination of unitary elements (to prove this, we may assume that z is Hermitian. In this case, we may as well restrict to the commutative von Neumann subalgebra of A generated by z. We may therefore assume that $A \simeq L^{\infty}(X)$ for some measure space X, in which case the desired result is not difficult). □

To find our trace, we would like to prove that there exists a fixed point for the action of $U(A)$ on the set $S(A)$. Here we need the following result from functional analysis, which we will prove in the next lecture:

Theorem 5 (Ryll-Nardzewski)**.** *Let M be a Banach space, let K be a convex subset of M which is compact with respect to the weak topology on M, and let G be a group of bounded operators on M which preserves K. Then there is an element of K which is fixed by the action of G.*

To apply this theorem to our situation, we will take M to be the predual of A: that is, the space of ultraweakly continuous linear functionals on M. Choose an ultraweakly continuous state $\phi \in S(A) \subseteq M$. Let $K_0 = \{\phi^u : u \in U(M)\}$, and let K denote the closed convex hull of K_0 in M (where we regard M as endowed with the weak topology). We will prove:

Proposition 6. *In the situation above, the set K is compact with respect to the weak topology on M.*

Assuming this result for the moment, it follows from Theorem 5 that the group $U(A)$ has a fixed point on K. Since $K \subseteq S(A)$, this fixed point is an ultraweakly continuous state on A. Lemma 4 then implies that this state is a trace, and the proof of Proposition 3 will be complete.

It remains to prove that the set K is weakly compact. Let $A^{\vee}$ denote the Banach space dual of A, endowed with the weak $*$-topology. We will regard M as a subspace of $A^{\vee}$. Let $\overline{K}$ denote the closure of K in $A^{\vee}$. Then $\overline{K}$ is a weak $*$-closed subset of the unit ball of $A^{\vee}$, hence compact for the weak $*$-topology. To prove the compactness of K, it will suffice to show that $K = \overline{K}$: that is, that every functional $\rho \in \overline{K}$ is automatically ultraweakly continuous.

Since each $\rho \in \overline{K}$ is a state, the ultraweak continuity of ρ is equivalent to complete additivity. Let $\{e_\alpha\}_{\alpha \in I}$ be a collection of mutually orthogonal projections of A. We have an inequality

$$\sum \rho(e_\alpha) \leq \rho(\sum e_\alpha),$$

and we wish to prove that it is an equality. In other words, we wish to show that for every positive real number ϵ, there exists a finite set $I_0 \subseteq I$ such that $\rho(\sum_{\alpha \notin I_0} e_\alpha) \leq \epsilon$.

Assume otherwise. Then, in particular, we have

$$\rho(\sum_{\alpha} e_\alpha) > \epsilon.$$

It follows that there exists $\rho_0 \in K_0$ such that $\rho_0(\sum_\alpha e_\alpha) > \epsilon$. Since ρ_0 is completely additive, there is a finite set $I_0 \subseteq I$ such that $\rho_0(\sum_{\alpha \in I_0} e_\alpha) > \epsilon$. Since $\rho(\sum_{\alpha \notin I_0} e_\alpha) > \epsilon$, we can find another functional $\rho_1 \in K_0$ such that $\rho_1(\sum_{\alpha \notin I_0} e_\alpha) > \epsilon$. Using the complete additivity of ρ_1, we can choose a finite subset $I_1 \subseteq I$ disjoint from I_0, such that $\rho_1(\sum_{\alpha \in I_1} e_\alpha) > \epsilon$. Since $\rho(\sum_{\alpha \notin I_0 \cup I_1} e_\alpha) > \epsilon$, we can find a functional $\rho_2 \in K_0$ such that $\rho_2(\sum_{\alpha \notin I_0 \cup I_1} e_\alpha) > \epsilon$. Using the complete additivity of ρ_2, we obtain a finite subset $I_2 \subseteq I$ disjoint from I_0 and I_1 such that $\rho_2(\sum_{\alpha \in I_2} e_\alpha) > \epsilon$, and so forth. Write $\rho_j = \phi^{u_j}$ for unitary elements $u_j \in U(A)$, and let $f_j = \sum_{\alpha \in I_j} e_\alpha$. We then have a sequence of unitary elements

$$u_0, u_1, u_2, \ldots \in U(A)$$

and a sequence of mutually orthogonal projections

$$f_0, f_1, f_2 \in A$$

such that $\phi(u_j^{-1} f_j u_j) > \epsilon$ for all j. Since ϕ is ultraweakly continuous, this contradicts the following claim:

Proposition 7. *Suppose we are given an infinite sequence $f_0, f_1, f_2, \ldots \in A$ of mutually orthogonal projections. Let $g_0, g_1, \ldots \in A$ be a collection of projections such that each g_i is conjugate to f_i (by a unitary element of A). If A is finite, then the sequence g_i converges to zero in the ultraweak topology (or even in the ultrastrong topology).*

Proof. We show that for every embedding $A \subseteq B(V)$, the sequence g_i converges strongly to zero in $B(V)$. Without loss of generality we may assume that $\sum f_i = 1$ (otherwise, set $e = 1 - \sum f_i$, and adjoint e to the beginning of both sequences). For each $m \geq 0$, let $V(m)$ denote the closed subspace of V generated by the subspaces $g_i V$ for $i \geq m$. We have a decreasing sequence of closed subspaces

$$V(0) \supseteq V(1) \supseteq V(2) \supseteq \cdots$$

of V. Let p_m denote the orthogonal projection from V onto $V(m)$. For each $v \in V$, we have $||g_m(v)|| \leq ||p_m(v)||$. It will therefore suffice to show that the operators p_m converge strongly to zero. Equivalently, it will suffice to show that the intersection $\bigcap_{m \geq 0} V(m)$ is zero.

If $m \leq n$, let $V(m,n)$ denote the subspace of V generated by $g_i V$ for $m \leq i \leq n$. Let W denote the orthogonal complement of $V(m, n-1)$ in $V(m,n)$. The composite map

$$g_n V \to V(m,n) \to W$$

is surjective and A'-linear. It follows that $W \leq g_n V \simeq f_n V$ as representations of A'. Applying this observation repeatedly, we obtain

$$V(m) \leq \bigoplus_{i \geq m} f_i V$$

. Taking orthogonal complements, we obtain

$$\bigoplus_{i<m} f_i V = (\bigoplus_{i \geq m} f_i V)^{\perp} \leq V(m)^{\perp}.$$

It follows that

$$(\bigcap_{m \geq 0} V(m))^{\perp} = \overline{\bigcup_{m \geq 0} V(m)^{\perp}}$$

is $\geq \bigoplus_{i<m} f_i V$ for each m, and is therefore $\geq \bigoplus f_i V = V$. Since V is finite as a representation of A', this implies that $(\bigcap_{m \geq 0} V(m))^{\perp} = V$: that is, $\bigcap_{m \geq 0} V(m) = 0$ as desired. □