Math 261y: von Neumann Algebras (Lecture 24)

October 28, 2011

In this lecture, we continue our study of faithful finite traces.

Proposition 1. *Let A be a factor, and let $\phi, \phi' : A \to \mathbf{C}$ be finite traces. Then $\phi = \phi'$.*

Proof. Write $A \subseteq B(V)$, so that V is finite when regarded as a module over the commutant A'. It follows that A' is a factor either of type I or type II. If A' has type I, then V is a finite sum of irreducible representations of A', so that $A = A''$ is isomorphic to a finite dimensional matrix ring. In this case, the uniqueness of the trace follows from linear algebra.

Let us assume that A' has type II. Let $R(A')_+$ denote the set of isomorphism classes of finite representations of A' and $R(A')$ the group obtained from $R(A)_+$ by adjoining inverses, so that $R(A')$ is isomorphic (as a linearly ordered abelian group) to the real numbers $\mathbb{R}$. There is a unique order-preserving isomorphism $\dim : R(A') \to \mathbb{R}$ which is normalized so that $\dim(V) = 1$. We will show that, for each projection operator $e \in A$, we have $\phi(e) = \dim(eV)$. The same argument gives $\phi'(e) = \dim(eV)$, so that ϕ and ϕ' coincide on projections of A. Since the linear span of the set of projections in A is norm-dense in A, it will follow that $\phi = \phi'$.

To prove that $\phi(e) = \dim(eV)$, we use ϕ to construct a map $d : R(A')_+ \to \mathbb{R}$. The construction proceeds as follows. Given a finite representation W of A', choose an integer n such that $\frac{1}{2^n}W \leq V$ (in the linearly ordered group $R(A')$). Then choose a isometric embedding $\rho : \frac{1}{2^n}W \hookrightarrow V$. The image of ρ has the form eV for some projection $e \in A$, and set $d(W) = 2^n \phi(e)$.

We claim that d is well-defined. We first show that the definition of $d(W)$ does not depend on the choice of embedding ρ. Suppose ρ is an embedding $\frac{1}{2^n}W \hookrightarrow V$ whose image has orthogonal complement V_0, and that $\rho' : \frac{1}{2^n}W \hookrightarrow V$ is an embedding with orthogonal complement V_1. Then

$$V_0 + \frac{1}{2^n}W = V = V_1 + \frac{1}{2^n}W$$

in the group $R(A')$, so that V_0 and V_1 are isomorphic (as representations of A'). It follows that there is an automorphism of V (as a representation of A') which carries $eV = \rho(\frac{1}{2^n}W)$ to $e'V = \rho'(\frac{1}{2^n}W)$ and V_0 to V_1. This automorphism is implemented by a unitary element $u \in A$ satisfying $e' = ueu^{-1}$. Then

$$\phi(e') = \phi(ueu^{-1}) = \phi(u^{-1}ue) = \phi(e).$$

We next show that $d(W)$ does not depend on the choice of n. Suppose that $\frac{1}{2^n}W \leq V$, and choose an embedding $\rho : \frac{1}{2^n}W \hookrightarrow V$ with image eV. Then $\frac{1}{2^{n+1}}W \leq V$, so we can choose another embedding $\rho' : \frac{1}{2^{n+1}}W \hookrightarrow V$ with image $e'V$. We wish to show that $2^n\phi(e) = 2^{n+1}\phi(e')$. We can factor $\frac{1}{2^n}W$ as a direct sum

$$\frac{1}{2^n}W \simeq \frac{1}{2^{n+1}}W \oplus \frac{1}{2^{n+1}}W,$$

so that ρ determines a pair of mutually orthogonal embeddings

$$\rho_-, \rho_+ : \frac{1}{2^{n+1}}W \hookrightarrow V$$

with images e_-V, e_+V, for some pair of mutually orthogonal projections $e_-, e_+ \in A$. The argument of the preceding paragraph shows that

$$\phi(e_-) = \phi(e') = \phi(e_+),$$

so that

$$\phi(e) = \phi(e_- + e_+) = \phi(e_-) + \phi(e_+) = 2\phi(e'),$$

as desired.

We now claim that the map $d : R(A')_+ \to \mathbb{R}$ is additive. Let W and W' be finite representations of A', and choose an integer n such that $\frac{1}{2^n}W + \frac{1}{2^n}W' \leq V$. Then we can find embeddings

$$\rho : \frac{1}{2^n}W \hookrightarrow V \qquad \rho' : \frac{1}{2^n}W' \hookrightarrow V$$

with orthogonal images eV and $e'V$. The sum of these is an embedding of $\frac{1}{2^n}(W + W')$ into V with image $(e + e')V$. Then

$$d(W + W') = 2^n\phi(e + e') = 2^n\phi(e) + 2^n\phi(e') = d(W) + d(W').$$

Since d is an additive map from $R(A)_+$ to the nonnegative real numbers, it extends uniquely to an order-preserving map $R(A) \to \mathbb{R}$. Taking $n = 0$ and $\rho : V \to V$ to be the identity, we deduce that $d(V) = 1$. It follows that d must coincide with our function $\dim : R(A) \to \mathbb{R}$, which proves that $\phi(e) = d(eV) = \dim(eV)$ for every projection $e \in A$. □

Corollary 2. *Let A be a factor. Then any finite trace $\phi : A \to \mathbf{C}$ is automatically ultraweakly continuous.*

Proof. Since A admits a faithful finite trace, it is automatically finite. In the next lecture we will prove every finite von Neumann algebra is a product of von Neumann algebras which admit faithful, ultraweakly continuous finite traces. Since A is a factor, we deduce that A admits an ultraweakly continuous trace ϕ'. Proposition 1 shows that $\phi = \phi'$, so that ϕ is ultraweakly continuous. □

Remark 3. The analogue of Corollary 2 fails if A is not a factor. Suppose that A is abelian. Then every state on A is tracial. For any commutative C^*-algebra A, there is a one-to-one correspondence between states on A and probability measures on $X = \operatorname{Spec} A$ (defined on the Σ-algebra of Baire sets). If A is a von Neumann algebra, then we have seen that a state is ultraweakly continuous if and only if the corresponding probability measure vanishes on all meager subsets of X.

Let A be a von Neumann algebra equipped with a faithful, ultraweakly continuous, finite trace ϕ. We let $L^2(A)$ denote the Hilbert space completion of A with respect to the inner product $(x, y) = \phi(y^*x)$. The left action of A on itself extends to an action of A on $L^2(A)$. Since ϕ is faithful, this gives an embedding $A \to B(L^2(A))$. We defer for the moment the proof of the following:

Proposition 4. *In the situation above, the action of A on $L^2(A)$ by right multiplication induces an isomorphism $A^{op} \to A'$, where A' is the commutant of A in $B(L^2(A))$.*

Corollary 5. *Let A be a finite factor. Then A has type I or II.*

Proof. We will show in the next lecture that A admits a faithful ultraweakly continuous finite trace ϕ. Let A' denote the commutant of A in $B(L^2(A))$. Then Proposition 4 gives an isomorphism $A^{op} \simeq A'$. Since A is finite, $L^2(A)$ is finite when regarded as a representation of A', so that A' has type I or II. It follows that A^{op} has type I or II, so that A has type I or II. □

The converse of Corollary 5 is false: a factor of type I or II need not be finite. For example, if $A = B(V)$ is a type I factor, then A is finite if and only if V is finite dimensional. We next establish an analogous result for type II factors.

First, we need a small digression on the classification of representations of type II factors. We have seen that if A is a type II factor, then the set of isomorphism classes of finite representations of A can be identified with $\mathbb{R}_{\geq 0}$. We now extend the picture to representations which are not finite.

Lemma 6. *Let A be a type II factor, let V be a nonzero finite representation of A, and let W be a representation of A which is not finite. Then W is isomorphic to $V^{\oplus I}$ for some infinite set I.*

Proof. Choose a maximal collection of isometric embeddings $\{\rho_\alpha : V \to W\}_{\alpha \in I}$ having mutually orthogonal images. The ρ_α induce an isometric embedding

$$f : V^{\oplus I} \to W.$$

Let W_0 be the orthogonal complement of the image of this embedding. By maximality, there cannot exist an isometric embedding of V into W_0. Thus $V \nleq W_0$. Since A is a factor, we have $W_0 < V$. In particular, V factors as an orthogonal direct sum $W_0 \oplus W_1$. If $V^{\oplus\infty}$ denotes a direct sum of countably many copies of V, we have

$$\begin{aligned} W_0 \oplus V^{\oplus\infty} &\simeq W_0 \oplus (W_1 \oplus V_0) \oplus (W_1 \oplus W_0) \oplus \cdots \\ &\simeq (W_0 \oplus W_1) \oplus (W_0 \oplus W_1) \oplus \cdots \\ &\simeq V^{\oplus\infty}. \end{aligned}$$

Since $W \simeq V^{\oplus I} \oplus W_0$ is not finite, the set I must be infinite. We conclude that

$$W \simeq V^{\oplus I} \oplus W_0 \simeq V^{\oplus I}.$$

□

Construction 7. Let V be a Hilbert space and let I be a set. Every bounded operator from $V^{\oplus I}$ to itself determines a matrix $[F_{i,j}]_{i,j \in I}$ whose entries are bounded operators from V to itself. If $A \subseteq B(V)$ is a von Neumann algebra, we let $M_{I \times I}(A)$ denote the subalgebra $B(V^{\oplus I})$ consisting of those elements whose matrix coefficients belong to A. This is a von Neumann algebra: it can be realized as the commutant of the diagonal action of A' on $V^{\oplus I}$.

Proposition 8. *Let A be a type II factor. Then one of the following possibilities holds:*

(1) *A is finite (in this case, we say that A has type II_1).*

(2) *There exists a finite type II factor B, an infinite set I, and an isomorphism $A \simeq M_{I,I}(B)$ (in this case, we say that A has type II_∞).*

Proof. Since A is type II, it has a nonzero finite representation V. Let A' denote the commutant of A in $B(V)$. If V is finite when regarded as an A'-module, then A is finite and we are done. Assume otherwise. The von Neumann algebra A' is finite (since V is finite as a representation of $A'' = A$), and therefore admits an ultraweakly continuous finite trace ϕ. Let $W = L^2(A')$ be the associated representation of A'. Then W is a finite representation of A'. It follows that $V \simeq W^{\oplus I}$ for some infinite set I. Then A is the commutant of the diagonal action of A' on $W^{\oplus I}$, which is given by $M_{I \times I}(B)$, where B is the commutant of A' in $B(W)$. Using Proposition 4, we get an equivalence $B \simeq A'^{op}$, which proves that B is finite. □