Math 261y: von Neumann Algebras (Lecture 23)

October 28, 2011

Let A be a factor. In the last lecture, we associated to A a linearly ordered abelian group $R(A)$: the collection $R(A)_+$ of nonnegative elements of $R(A)$ can be identified with the set of isomorphism classes of finite representations of A. Our first goal in this lecture is to determine the possible structures on $R(A)$. There are three cases to consider:

(a) The group $R(A)$ is trivial: that is, A has no nontrivial finite representations. In this case, we say that $R(A)$ is a *type III factor*.

(b) There exists a smallest positive element of $R(A)$. This element corresponds to a representation V. Let $W \subset V$ be a proper A-submodule. Since V is finite, we must have $W < V$. Since V is a least positive element of $R(A)$, we have $W \simeq 0$. This proves that V is irreducible, so that the von Neumann algebra A is type I.

(c) Suppose that $R(A)$ is nontrivial, but has no least positive element. Fix a positive element $V \in R(A)$. We define a map $\phi : R(A) \to \mathbb{R}$ as follows. Given $W \in R(A)$, let

$$\mathbf{Q}_{\leq W} = \{\frac{p}{q} : (q > 0) \wedge pV \leq qW\} \qquad \mathbf{Q}_{<W} = \mathbf{Q}_{\leq W} = \{\frac{p}{q} : (q > 0) \wedge pV < qW\}.$$

Since V is positive, the Archimedean property of $R(A)$ implies that

$$-nV < W < nV$$

for n sufficiently large. It follows that the sets $\mathbf{Q}_{<W}$ and $\mathbf{Q}_{\leq W}$ are nonempty and bounded above. Since they differ by at most a single rational number, they have the same supremum, which we will denote by $\phi(W)$. We have

$$\begin{aligned} -\phi(W) &= -\sup \mathbf{Q}_{\leq W} \\ &= \inf(-1)\,\mathbf{Q}_{\leq W} \\ &= \inf(\mathbf{Q} \,/\, \mathbf{Q}_{<-W}) \\ &= \sup \mathbf{Q}_{<-W} \\ &= \phi(-W). \end{aligned}$$

It is clear that ϕ is monotone: if $W \leq W'$, then $\mathbf{Q}_{\leq W} \subseteq \mathbf{Q}_{\leq W'}$ so that $\phi(W) \leq \phi(W')$. We next show that ϕ is a group homomorphism. Let $W, W' \in R(A)$. If $\frac{p}{q} \in \mathbf{Q}_{\leq W}$ and $\frac{p'}{q'} \in \mathbf{Q}_{\leq W'}$, then we have

$$pV \leq qW \qquad p'V \leq q'W'$$

so

$$pq'V \leq qq'W \qquad p'qV \leq qq'W'$$
$$(pq' + p'q)V \leq qq'(W + W')$$

$$\frac{p}{q} + \frac{p'}{q'} \in \mathbf{Q}_{\leq W+W'}.$$

This proves that $\mathbf{Q}_{\leq W} + \mathbf{Q}_{\leq W'} \subseteq \mathbf{Q}_{\leq W+W'}$ so that $\phi(W)+\phi(W') \leq \phi(W+W')$. The reverse inequality then follows by applying the same arguments to $-W$ and $-W'$.

We now claim that ϕ is injective. Assume otherwise; then there exists a positive element $W \in R(A)$ such that $\phi(W) = 0$. Using the Archimedean property, we deduce that there exists an integer n such that $V < nW$. Then $\frac{1}{n} \in \mathbf{Q}_W$, contradicting the assumption that $\phi(W) = 0$.

It remains to prove that ϕ is surjective. Let us denote the image of ϕ by $K \subseteq \mathbb{R}$. We wish to show that $K = \mathbb{R}$. Since K is a nontrivial subgroup of $\mathbb{R}$ with no least element, it is dense in $\mathbb{R}$. It will therefore suffice to show that K is closed. Let $t \in \overline{K}$; we wish to show that $x \in K$. We can write t as the limit of a sequence of elements $t_0 = t_1, t_2, \ldots \in K$ which is either increasing or decreasing; we will assume without loss of generality that the sequence is increasing. Then we can write $t_{i+1} - t_i = \phi(W_i)$ for some finite representations W_i of A. We will show that $W = \bigoplus W_i$ is a finite representation of A and that $x = t_0 + \phi(W)$ belongs to K. To prove the second claim, it will suffice to show that $t_0 + \phi(W) \geq r$ for every element $r \in K$ such that $r \geq x$. Writing $r - t_0 = \phi(U)$, we are reduced to proving that $W \leq U$ (which simultaneously proves the finiteness of W).

Note that we have $\sum \phi(W_i) \leq \phi(U)$. In particular $\phi(W_0) \leq \phi(U)$, so there exists an embedding $f_0 : W_0 \hookrightarrow U$. Denote its orthogonal complement by U_1; then $\phi(W_0) + \phi(W_1) \leq \phi(U)$ implies that $W_1 \leq U_1$ so we can choose an embedding $f_1 : W_1 \hookrightarrow U_1 \subseteq U$. Proceeding in this way, we obtain a collection of embeddings $f_i : W_i \rightarrow U$ with mutually disjoint images, which gives an isometric embedding $\bigoplus W_i \hookrightarrow U$.

We say that a factor A is *type II* if the third case occurs: that is, if A has finite representations but no irreducible representations.

Definition 1. Let $A \subseteq B(V)$ be a von Neumann algebra with commutant A'. We will say that A is *finite* if V is finite when regarded as an A'-module.

Remark 2. In the situation of Definition 1, there is a bijective correspondence between closed A'-submodules of V and projections in A. Moreover, if $e \in A$ is a projection, then an isomorphism of V with eV (as A'-modules) can be identified with an operator $u \in A$ satisfying $uu^* = e$, $u^*u = 1$. Note that the second condition implies that

$$(uu^*)(uu^* = uu^*,$$

so that uu^* is automatically a projection. It follows that A is finite if and only if the following condition is satisfied:

($*$) For every element $u \in A$ satisfying $u^*u = 1$, we have $u^*u = 1$.

In particular, this condition is intrinsic to A: it does not depend on the embedding $A \subseteq B(V)$.

We now study a mechanism for proving that a von Neumann algebra is finite.

Proposition 3. *Let A be a von Neumann algebra and let $\phi : A \rightarrow \mathbf{C}$ be a state. The following conditions are equivalent:*

(1) *For every $x, y \in A$, we have $\phi(xy) = \phi(yx)$.*

(2) *For every Hermitian element $h \in A$ and every unitary element $u \in A$, we have $\phi(uhu^{-1}) = \phi(h)$.*

Proof. To show that (1) $\Rightarrow$ (2), take $x = uh$ and $y = u^{-1}$. For the converse, suppose that (2) is satisfied. Then every element $h \in A$ satisfies $\phi(uhu^{-1}) = \phi(h)$ (since the Hermitian elements generate A as a $\mathbf{C}$-vector space). Taking $h = xu$, we obtain $\phi(ux) = \phi(xu)$ for each $x \in A$ and each unitary element $u \in A$. To prove (1), it suffices to show that A is the $\mathbf{C}$-linear span of its unitary elements. It suffices to prove that every

Hermitian element $y \in A$ belongs to this span. Replacing A by the abelian von Neumann algebra generated by y, we can reduce to the case where $A = L^\infty(X)$, in which case the desired result follows from elementary considerations. □

Definition 4. Let A be a von Neumann algebra and let $\phi : A \to \mathbf{C}$ be a state. We say that ϕ is *tracial* if it satisfies the equivalent conditions of 3. In this case, we also say that ϕ is a *finite trace*. We say that ϕ is *faithful* if, for every positive element $x \in A$, either $x = 0$ or $\phi(x) > 0$.

Proposition 5. *Let A be a von Neumann algebra. If A admits a faithful finite trace, then A is finite.*

Proof. Let $u \in A$ be a partial isometry satisfying $u^*u = 1$; we wish to show that $uu^* = 1$. Write $e = uu^*$. Then e is a projection, and we have $\phi(e) = \phi(uu^*) = \phi(u^*u) = \phi(1)$. Thus $\phi(1 - e) = 0$. Since $1 - e$ is positive and ϕ is faithful, this implies that $1 - e = 0$, so that $e = uu^* = 1$ as desired. □

We have the following converse:

Theorem 6. *Let A be a finite von Neumann algebra. Then A can be written as a (von Neumann algebra) product $\prod A_\alpha$, where each A_α admits a faithful finite trace which is ultraweakly continuous.*

Remark 7. From the characterization given in Remark 2, it is easy to see that a product of finite von Neumann algebras is itself finite. Thus the criterion of Theorem 6 is both necessary and sufficient.

Remark 8. If A is a factor, then one can prove that every tracial state is automatically ultraweakly continuous. We will not use this fact.

Here is a rough idea of why Theorem 6 should be true. Assume for simplicity that $A \subseteq B(V)$ is a factor, so that V is finite when regarded as a representation of A'. There is a unique order-preserving isomorphism $\rho : R(A') \to \mathbb{R}$ such that $\rho(V) = 1$. We can think of ρ as a function which assigns a "dimension" to each finite representation of A'. In particular, if $e \in A$ is a projection, then eV is a closed A'-submodule of V, hence finite as a representation of A'. It therefore has a well-defined dimension $\rho(eA)$. We would like to define a tracial state $\phi : A \to \mathbf{C}$ by the formula

$$\phi(e) = \rho(eA).$$

Unfortunately, this formula only makes sense when e is a projection: to get a state, we need to define ϕ on arbitrary elements of A. However, since A is generated by its projections, any (ultraweakly continuous) state ϕ is determined by its restriction to the projections. We might then hope to show that the above prescription extends uniquely to a state $\phi : A \to \mathbf{C}$. We postpone giving a real proof for the moment; we will return to the matter next week.

Let's explore some of the consequences of having a faithful finite trace. Recall that for any state $\phi : A \to \mathbf{C}$, we can associate an inner product on A, given by $(x, y) = \phi(y^*x)$. We then have

$$(zx, y) = \phi(y^*zx) = \phi((z^*y)^*x) = (x, z^*y).$$

In other words, the action of A on itself by left multiplication is a $*$-homomorphism. However, it is not at all obvious that the same is true for right multiplication: we have

$$(xz, y) = \phi(y^*xz) \qquad (x, yz^*) = \phi(zy^*x).$$

To say that these expressions are the same (for all x, y, and z) is to say that the right action of A on itself is via $*$-homomorphisms. If we let V_ϕ denote the Hilbert space completion of A with respect to the inner product $(,)$, this implies that the right action of A on itself extends to a right action of A on V_ϕ (note that if $z \in A$ has norm ≤ 1, then we can write $1 = z^*z + z'^*z'$ for some $z' \in A$. If we let r_z and $r_{z'}$ denote right multiplication by z and z', we get $r_z^*r_z + r_{z'}^*r_z = 1$, which forces r_z to have operator norm ≤ 1).

Proposition 9. *Let A be a von Neumann algebra, let $\phi : A \to \mathbf{C}$ be a faithful finite trace whichi is ultraweakly continuous, and let V_ϕ denote the Hilbert space associated to ϕ. The left action of A on itself induces an embedding $\rho : A \hookrightarrow B(V_\phi)$. Let A' denote its commutant. Then the right action of A on V_ϕ induces an isomorphism $\rho' : A^{op} \to A'$.*

We will give the proof of this (and deduce some consequences) in the next lecture.