Math 261y: von Neumann Algebras (Lecture 22)

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Fix a von Neumann algebra A. In this lecture, we will study the category $\mathrm{Rep}(A)$ of (ultraweakly continuous) representations of A. Given a pair of representations V and W, we will write $V \leq W$ if V appears as an (orthogonal) direct summand of W. This relation is evidently reflexive and transitive. It also enjoys the following assymmetry property:

Proposition 1. *Let V and W be representations of a von Neumann algebra A. If $V \leq W$ and $W \leq V$, then $V \simeq W$.*

Proof. Since $W \leq V$, we can identify W with a closed subspace of V. The assumption that $V \leq W$ implies that there exists an isometric A-linear embedding $f : V \to W$. Set $V_0 = V$, $V_1 = W$, and $V_i = f(V_{i-2})$ for $i \geq 2$. Then

$$V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots$$

Let U_i denote the orthogonal complement of V_{i+1} in V_i, so that the U_i are mutually orthogonal subspaces of V. Let $V_\infty = \bigcap V_i$, so that V_∞ is the orthogonal complement of $\bigoplus U_i$. Then we can recover V as the direct sum of V_∞ with $\bigoplus_{i \geq 0} U_i$, and W as the direct sum of V_∞ with $\bigoplus_{i \geq 1} U_i$. It will therefore suffice to show that

$$\bigoplus_{i \geq 0} U_i \simeq \bigoplus_{i \geq 1} U_i.$$

This is clear, since the map f induces isomorphisms $U_i \to U_{i+2}$ for each i. □

By virtue of Proposition 1, we can regard the collection of isomorphism classes of representations of A as partially ordered via $\leq$.

Remark 2. Let V and W be representations of a von Neumann algebra A. Suppose there exists a bounded A-linear map $f : V \to W$ which is injective (but not necessarily an isometry). We can then endow V with a new inner product, given by the inner product on W. This inner product has the form (v, Hw) for some positive operator $H \in B(V)$. Since f commutes with the action of A, the operator H commutes with the action of A. It follows that f factors as a composition

$$V \stackrel{\sqrt{H}}{\to} V \stackrel{u}{\to} W$$

where u is an isometry, so that $V \leq W$.

Proposition 3. *Let A be a von Neumann algebra. The following conditions are equivalent:*

(1) *A is a factor.*

(2) *Every nonzero representation of A is faithful.*

(3) *The partial ordering $\leq$ on isomorphism classes of objects of $\mathrm{Rep}(A)$ is a linear ordering.*

Proof. If V is a nonzero representation of A, then the kernel of the map $A \to B(V)$ is an ultraweakly closed two-sided ideal of A, hence of the form eA for some central projection $e \in A$. If A is a factor, we conclude that either $e = 0$ (in which case V is faithful) or $e = 1$ (in which case $V = 0$). This proves that (1) $\Rightarrow$ (2).

We next show that (3) $\Rightarrow$ (1). Suppose that A is not a factor. Then there exists a central projection $e \in A$ such that $e \notin \{0, 1\}$. Let V be a faithful representation of A, so that V factors as an orthogonal direct sum $eV \oplus (1-e)V$. Since e vanishes on $(1-e)V$ and acts by the identity on eV, we cannot have $eV \leq (1-e)V$ or $(1-e)V \leq eV$.

We complete the proof by showing that (2) $\Rightarrow$ (3). Let V and W be representations of A. Let X be the set of all triples (V_0, W_0, ϕ) where V_0 is a closed submodule of V, W_0 is a closed submodule of W, and ϕ is an isometric A-linear isomorphism of V_0 with W_0. Let us say that a pair of elements (V_0, W_0, ϕ) and (V_1, W_1, ψ) are *orthogonal* if $V_0 \perp V_1$ and $W_0 \perp W_1$. Using Zorn's lemma, we deduce that there exists a maximal collection of mutually orthogonal elements $S = \{(V_\alpha, W_\alpha, \phi_\alpha)\}$ of X. Let V' denote the orthogonal complement of the sum $\bigoplus V_\alpha \subseteq V$ and W' the orthogonal complement of the sum $\bigoplus W_\alpha \subseteq W$. If $V' = 0$, then the maps ϕ_α determine an isometric embedding of V into W, so that $V \leq W$. Similarly, if $W' = 0$, then $W \leq V$. Let us therefore assume that $V', W' \neq 0$. Assumption (2) implies that V' is a faithful representation of A. It follows that every representation of A (and in particular W') can be written as a direct summand of a direct sum of copies of V'. In particular, there exists a nonzero map $\rho : V' \to W'$, hence an injection $\ker(\rho)^\perp \hookrightarrow W'$. Using Remark 2 we obtain $\ker(\rho)^\perp \leq W'$, so there exists a submodule $W_0' \subseteq W$ and an isomorphism $\phi : \ker(\rho)^\perp \to W_0'$. This contradicts the maximality of S. □

Definition 4. Let A be a von Neumann algebra and let V be a representation of A. We will say that V is *irreducible* if $V \neq 0$ and for every closed A-submodule $W \subseteq V$, we have either $W = 0$ or $W = V$. We will say that V is *finite* if, for every closed A-submodule $W \subseteq V$ which is isometrically isomorphic to V (as a repersentation of A), we have $W = V$.

Warning 5. The terminology of Definition 4 may be nonstandard. The standard terminology is to refer to a projection $p \in A \subseteq B(V)$ as *finite* if the subspace $pV \subseteq V$ is finite when regarded as a representation of A'.

Let V be a representation of A, and let A' denote the commutant of the image of A in $B(V)$. Then there is a bijective correspondence between projections in A' and closed A-submodules of V. Since A' is generated by the projections contained in A', we see that V is irreducible if and only if $A' \simeq \mathbf{C}$. From this we conclude:

Proposition 6. *Let A be a von Neumann algebra. Then A is a type I factor if and only if A has a faithful irreducible representation. If A is a factor, then A is type I if and only if it has an irreducible representation.*

Let us now study the class of finite representations.

Example 7. Let A be a type I factor. Then $\mathrm{Rep}(A)$ is equivalent to the category of Hilbert spaces. It follows that a representation V of A is finite if and only if it can be written as a direct sum of finitely many irreducible representations of A.

Proposition 8. *Let A be a von Neumann algebra and let V be a representation of A. If V is finite, then any closed A-submodule of V is finite.*

Proof. Let $W \subseteq V$ be a closed A-submodule and suppose that W is isomorphic to W', for some closed A-submodule $W' \subseteq W$. Then $V = W \oplus W^\perp$ is isomorphic to $W' \oplus W^\perp$. Since V is finite, we conclude that $V = W' \oplus W^\perp$, so that $W' = W$. □

Proposition 9. *Let A be a factor, and let V and W be finite representations of A. Then the direct sum $V \oplus W$ is finite.*

Lemma 10. *Let A be a von Neumann algebra and let V be a representation of A. If V is infinite, then there exists a pair of closed A-submodules $U, U' \subseteq A$ which are infinite and orthogonal to one another.*

Proof. Since V is infinite, there exists an A-linear isometry f from V to a proper submodule of itself. For $n \geq 0$, set $V_n = f^n(V)$, so that

$$V = V_0 \supset V_1 \supset V_2 \supset \cdots$$

Let W_i denote the orthogonal complement of V_{i+1} in V_i. Then f induces isometric isomorphisms $W_i \to W_{i+1}$. Take $U = \bigoplus_k W_{2k}$ and $U' = \bigoplus_k W_{2k+1}$. Then f^2 induces isomorphisms from U and U' to proper submodules of themselves (so that U and U' are infinite). □

Proof of Proposition 9. Assume that V and W are finite, and suppose that $V \oplus W$ is infinite. Then there exist closed A-submodules $U, U' \subseteq V \oplus W$ which are orthogonal and infinite. According to Proposition 3, we have either $V \cap U \leq W \cap U'$ or $W \cap U' \leq V \cap U$. Without loss of generality, we may assume that $V \cap U \leq W \cap U'$. Then

$$U = (V \cap U) \oplus (U \cap (U \cap V)^{\perp}).$$

Let W' denote the image of $U \cap (U \cap V)^{\perp}$ in W, so that $U \cap ((U \cap V)^{\perp} \leq W'$ by virtue of Remark 2. We therefore get

$$U \leq (W \cap U') \oplus W'.$$

Since U' is orthogonal to U, $W \cap U'$ is orthogonal to W', so the right hand side can be identified with a closed subspace of W. Since W is finite, Proposition 8 implies that U is finite, contradicting our assumption that U is infinite. □

Let A be a factor. We let $R_+(A)$ denote the collection of all isomorphism classes of finite representations of A. The construction

$$V, W \mapsto V \oplus W$$

endows $R_+(A)$ with the structure of a commutative monoid under addition.

Proposition 11. *Let A be a factor. Then the monoid $R_+(A)$ is cancellative. That is, U, V, and W are finite representations of A and $U \oplus W \simeq V \oplus W$, then $U \simeq V$.*

Proof. Without loss of generality, we may assume that $U \leq V$, so that $V \simeq K \oplus U$ for some $K \in \mathrm{Rep}(A)$.Then

$$V \oplus W \simeq K \oplus U \oplus W \simeq K \oplus (V \oplus W).$$

Since $V \oplus W$ is finite (Proposition 9), we conclude that $K \simeq 0$, so that $U \simeq V$. □

If A is a factor, let $R(A)$ denote the group obtained from $R_+(A)$ by formally adjoining inverses. It follows from Proposition 11 that the map $R_+(A) \to R(A)$ is injective. We will identify $R(A)$ with its image in $R_+(A)$.

Proposition 12. *Let A be a factor. Then $R(A) = R_+(A) \cup -R_+(A)$. Moreover, the intersection of $R_+(A)$ with $-R_+(A)$ consists only of the zero element (corresponding to the trivial representation of A). Consequently, $R(A)$ inherits the structure of a linearly ordered abelian group A, where $x \leq y$ if and only if $y - x \in R_+(A)$.*

Proof. If $x, y \in R_+(A)$ with $x + y = 0$, then x and y are both zero (since a direct sum of representations is zero if and only if each summand is zero). Note that an arbitrary element of $R(A)$ can be written as a difference $x - y$, where $x, y \in R_+(A)$. Then x and y are equivalence classes of representations $V, W \in \mathrm{Rep}(A)$. We have either $V \leq W$ or $W \leq V$. Assume without loss of generality that $W \leq V$. Then $V \simeq W \oplus W'$, so $x = y + y'$ in $R_+(A)$ and therefore $x - y = y' \in R_+(A)$. □

Remark 13. When restricted to the subset $R_+(A) \subseteq R(A)$, the linear ordering of Proposition 12 agrees with the linear ordering on representations introduced at the beginning of this lecture.

Proposition 14. *Let A be a factor. Then the linearly ordered abelian group $R(A)$ is Archimedean. That is, if x and y are positive elements of $R(A)$, then $x < ny$ for $n \gg 0$.*

Proof. The elements x and y correspond to isomorphism classes of finite representations $V, W \in \text{Rep}(A)$. Suppose for a contradiction that we have $ny \leq x$ for every integer n. Taking $n = 1$, we obtain an isometric embedding $f_1 : V \to W$. Let W_1 be the orthogonal complement of the image of f_1. Then W_1 represents $x - y \in R(A)$. Taking $n = 2$, we get $y \leq x - y$, so there exists an isometric embedding $f_2 : V \to W_1 \subseteq W$ Continuing in this way, we obtain an infinite collection of isometric embeddings $f_i : V \to W$ with mutually orthogonal images. It follows that $V^{\oplus\infty}$ appears as a direct summand of W. Since $V \neq 0$, $V^{\oplus\infty}$ is infinite, so that W is infinite, contrary to our assumptions. □

Recall that if G is a linearly ordered abelian group satisfying the Archimedean property, then there exists an order-preserving embedding $G \hookrightarrow \mathbb{R}$. In particular, we obtain an embedding $R(A) \hookrightarrow \mathbb{R}$.

We will prove the following result in the next lecture:

Proposition 15. *Let A be a factor. Then exactly one of the following conditions holds:*

- (a) *The group $R(A)$ is isomorphic (as a linearly ordered group) to* $\mathbf{Z}$.
- (b) *The group $R(A)$ is isomorphic (as a linearly ordered group) to* $\mathbb{R}$.
- (c) *The group $R(A)$ is trivial.*

Remark 16. The group $R(A)$ is isomorphic to $\mathbf{Z}$ if and only if there is a smallest positive element of $R(A)$. The existence of such an element is equivalent to the existence of an irreducible representation of A. Consequently, case (a) of Proposition 15 applies precisely when A is a type I factor.

Definition 17. Let A be a factor. We say that A is *type II* if case (b) of Proposition 15 holds, and *type III* if case (c) of Proposition 15 holds. That is, A is type II if it has finite representations but no irreducible representations, and type III if it has no finite representations.