Math 261y: von Neumann Algebras (Lecture 21)

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Let A be a von Neumann algebra. We let $\mathrm{Rep}(A)$ denote the category of ultraweakly continuous (Hilbert) representations of A. The category $\mathrm{Rep}(A)$ has the following structures:

(a) For every morphism $f : V \to W$ in $\mathrm{Rep}(A)$, there is another morphism $f^* : W \to V$, given (at the level of Hilbert spaces) by the adjoint of f.

(b) For every collection of objects V_α in $\mathrm{Rep}(A)$, there is another object $\bigoplus V_\alpha \in \mathrm{Rep}(A)$, given at the level of Hilbert spaces by the orthogonal direct sum of the V_α (note that this is *not* a categorical coproduct of the V_α: an arbitrary collection of bounded A-linear maps $V_\alpha \to W$ need not extend to an A-linear map $\bigoplus V_\alpha \to W$.)

We let Hilb denote the category of Hilbert spaces. We will identify Hilb with $\mathrm{Rep}(\mathbf{C})$, so that Hilb has the structures mentioned above.

Definition 1. Let A be a von Neumann algebra. A functor $F : \mathrm{Rep}(A) \to \mathrm{Hilb}$ is *completely additive* if it is $\mathbf{C}$-linear and compatible with the structures (a) and (b) described above. That is, if F satisfies the following conditions:

(a') For every bounded A-linear map $f : V \to W$, we have $F(f^*) = F(f)^*$ as operators from $F(W)$ to $F(V)$.

(b') For every collection of objects $V_\alpha \in \mathrm{Rep}(A)$, the collection of maps

$$F(V_\alpha) \to F(\bigoplus V_\alpha)$$

exhibits $F(\bigoplus V_\alpha)$ as an orthogonal direct sum of the $F(V_\alpha)$.

(c') For every pair of objects $V, W \in \mathrm{Rep}(A)$, F induces a $\mathbf{C}$-linear map

$$\mathrm{Hom}_{\mathrm{Rep}(A)}(V, W) \to \mathrm{Hom}_{\mathrm{Hilb}}(F(V), F(W)).$$

(we can actually prove that this map is $\mathbb{R}$-linear using (a') and (b'), but we will not need this).

Our first goal in this lecture is to classify all of the completely additive functors $\mathrm{Rep}(A) \to \mathrm{Hilb}$. To this end, let us fix an embedding $A \hookrightarrow B(V)$ for some Hilbert space V, so that we can identify V with a representation of A. Let A' denote the commutant of A in $B(V)$. Every $f \in A'$ can be identified with a morphism from V to itself in $\mathrm{Rep}(A)$. If $F : \mathrm{Rep}(A) \to \mathrm{Hilb}$ is a completely additive functor, then $F(f)$ is a bounded operator on the Hilbert space $F(V)$. The construction

$$f \mapsto F(f)$$

determines a map

$$A' \to B(F(V)).$$

It follows from (c') that this is a map of $\mathbf{C}$-vector spaces, from functoriality that it is a map of $\mathbf{C}$-algebras, and from (a') that it is a map of $*$-algebras. In fact, we can do better:

Lemma 2. *In the above situation, the map $\rho : A' \to B(F(V))$ is ultraweakly continuous.*

Proof. We will show that ρ is completely additive. Let p_α be a collection of mutually orthogonal projections in A', having images $V_\alpha \subseteq V$. Then the sum of the projections p_α can be identified with the composite map

$$V \xrightarrow{\{p_\alpha\}} \bigoplus V_\alpha \to V.$$

Applying F, we see that $\rho(\sum p_\alpha)$ is given by the composite map

$$F(V) \xrightarrow{\{F(p_\alpha)\}} F(\bigoplus V_\alpha) \to F(V)$$

Using the complete additivity of F, this is given by

$$F(V) \xrightarrow{\{F(p_\alpha)\}} \bigoplus F(V_\alpha) \to F(V),$$

and is therefore the sum of the mutually orthogonal projections $F(p_\alpha)$. ☐

Let $\mathrm{Fun}^c(\mathrm{Rep}(A), \mathrm{Hilb})$ denote the category of completely additive functors from $\mathrm{Rep}(A)$ to Hilb. The construction above shows that if $A \subseteq B(V)$, then evaluation on V determines a functor

$$\theta : \mathrm{Fun}^c(\mathrm{Rep}(A), \mathrm{Hilb}) \to \mathrm{Rep}(A').$$

Proposition 3. *The functor θ is an equivalence of categories.*

Proof. We define a category $\mathrm{Rep}'(A)$ as follows:

- The objects of $\mathrm{Rep}'(A)$ are sets I. Given a set I, we let $V^{\oplus I}$ denote an orthogonal direct sum of copies of V, indexed by I.
- Given a pair of sets I and J, a map from I to J is a bounded operator $V^{\oplus I} \to V^{\oplus J}$ which commutes with the action of A.

The construction $I \mapsto V^{\oplus I}$ determines a fully faithful embedding of $\mathrm{Rep}'(A)$ into $\mathrm{Rep}(A)$. We have seen that every object of $\mathrm{Rep}(A)$ is a direct summand of an object of the form $V^{\oplus I}$. In categorical terms, this means that $\mathrm{Rep}(A)$ can be described as the *idempotent completion* of $\mathrm{Rep}'(A)$. The category Hilb is idempotent complete, so the category of functors from $\mathrm{Rep}(A)$ to Hilb is equivalent to the category of functors from $\mathrm{Rep}'(A)$ to Hilb. In particular, we can identify $\mathrm{Fun}^c(\mathrm{Rep}(A), \mathrm{Hilb})$ with a subcategory $\mathrm{Fun}^c(\mathrm{Rep}'(A), \mathrm{Hilb}) \subseteq \mathrm{Fun}(\mathrm{Rep}'(A), \mathrm{Hilb})$ spanned by those functors which satisfy the obvious analogues of the conditions listed in Definition 1.

Let us now explicitly describe the inverse of θ. Let W be a representation of A'. We will associate to W a functor

$$\phi_W : \mathrm{Rep}'(A) \to \mathrm{Hilb}$$

given on objects by

$$I \mapsto W^{\oplus I}.$$

The hard part is to define ϕ_W on morphisms. We would like to say the following: a map from I to J in $\mathrm{Rep}'(A)$ is given by an I-by-J matrix $M_{i,j}$ with coefficients in the von Neumann algebra A'. Then $M_{i,j}$ should determine a map from $W^{\oplus I}$ to $W^{\oplus J}$. If I and J are finite, this is clear. In the general case, some analysis is involved.

Let U be a representation of A'. Let us say that an I-by-J matrix with coefficients in A' (that is, a map $I \times J \to A'$) is *U-good* if it determines a bounded operator from $U^{\oplus I}$ to $U^{\oplus J}$. Note that we can identify $\mathrm{Hom}_{\mathrm{Rep}'(A)}(I, J)$ with the collection of V-good matrices in $A'^{I \times J}$. To guarantee that our description of ϕ_W above is well-defined, we want to know that every V-good matrix is also W-good. As a representation of A', we can realize W as a direct summand of $V^{\oplus K}$ for some set K. It will therefore suffice to show that every V-good matrix is also $V^{\oplus K}$-good. This is clear (an orthogonal direct sum of uniformly bounded operators is itself a bounded operator), so that ϕ_W is well-defined. It is now easy to see that the construction $W \mapsto \phi_W$ is an inverse to the $\mathrm{Fun}^c(\mathrm{Rep}'(A), \mathrm{Hilb}) \to \mathrm{Rep}(A')$ given by evaluation on a one-element set. ☐

Now suppose that we are given a pair of von Neumann algebras A and B, with $A \subseteq B(V)$. We can then speak of completely additive functors from $\mathrm{Rep}(A)$ to $\mathrm{Rep}(B)$; we will denote the category of such functors by $\mathrm{Fun}^c(\mathrm{Rep}(A), \mathrm{Rep}(B))$. Note that the following data are equivalent:

- Completely additive functors from $\mathrm{Rep}(A)$ to $\mathrm{Rep}(B)$.
- Completely additive functors F from $\mathrm{Rep}(A)$ to Hilb, together with an (ultraweakly continuous) representation of B on $F(W)$ for each $W \in \mathrm{Rep}(A)$, depending functorially on W.
- Completely additive functors F from $\mathrm{Rep}(A)$ to Hilb equipped with an (algebraic) action of the algebra B, such that the induced action of B on $F(V)$ is a von Neumann algebra representation. (Since every representation of A appears as a direct summand of a direct sum of copies of V, this implies that the action of B on each $F(W)$ is a von Neumann algebra representation).
- Representations of the von Neumann algebra A' on a Hilbert space H, equipped with an action of B which makes H into a von Neumann algebra representation of B.

This motivates the following definition:

Definition 4. Let A and B be von Neumann algebras. An A-B *bimodule* is a Hilbert space V equipped with (ultraweakly continuous) actions of A and B^{op} which commute with one another.

Our analysis proves the following:

Proposition 5. *Let A and B be von Neumann algebras, and suppose that A is given as a von Neumann subalgebra of $B(V)$ for some Hilbert space V. Then evaluation on V induces an equivalence from the category* $\mathrm{Fun}^c(\mathrm{Rep}(A), \mathrm{Rep}(B))$ *to the category of B-A'^{op} bimodules.*

Remark 6 (Connes Fusion)**.** Suppose we are given three von Neumann algebras A, B, and C, with $A \subseteq B(V)$ and $B \subseteq B(W)$. Proposition 5 allows us to identify the categories $\mathrm{Fun}^c(\mathrm{Rep}(A), \mathrm{Rep}(B))$ and $\mathrm{Fun}^c(\mathrm{Rep}(B), \mathrm{Rep}(C))$ with the categories of B-A'^{op} bimodules and C-B'^{op} bimodules, respectively. There is an evident composition functor

$$\mathrm{Fun}^c(\mathrm{Rep}(A), \mathrm{Rep}(B)) \times \mathrm{Fun}^c(\mathrm{Rep}(B), \mathrm{Rep}(C)) \to \mathrm{Fun}^c(\mathrm{Rep}(A), \mathrm{Rep}(C)).$$

We can think of this as a sort of tensor product which takes a B-A'^{op} bimodule and a C-B'^{op} bimodule and outputs a C-A'^{op} bimodule. This operation is a version of *Connes fusion*.

As described above, the fusion construction is dependent on choices of realization $A \subseteq B(V)$ and $B \subseteq B(W)$. We will later see that every von Neumann algebra A has a *canonical* realization on a Hilbert space $L^2(A)$. Moreover, the commutant of A in $B(L^2(A))$ can be identified with A^{op}. Then Proposition 5 gives an identification of $\mathrm{Fun}^c(\mathrm{Rep}(A), \mathrm{Rep}(B))$ with the category of B-A bimodules, and Connes fusion is an operation which takes an B-A-bimodule H and C-B bimodule H' and returns a C-A bimodule

$$H' \boxtimes_B H.$$

Definition 7. Let A and B be von Neumann algebras. A *Morita equivalence* between A and B is a completely additive functor from $\mathrm{Rep}(A)$ to $\mathrm{Rep}(B)$ which is an equivalence of categories. We will say that A and B are *Morita equivalent* if there is a Morita equivalence from A to B.

A property of von Neumann algebras A is invariant under Morita equivalence if and only if it can be described purely in terms of the category $\mathrm{Rep}(A)$.

Example 8. Let $A \subseteq B(V)$ be a von Neumann algebra. Proposition 5 implies that the category of completely additive functors from $\mathrm{Rep}(A)$ to itself is equivalent to the category of A-A'^{op} bimodules. In particular, the identity functor corresponds to the bimodule V. Consequently, the algebra of *endomorphisms* of the identity functor of $\mathrm{Rep}(A)$ is given by the collection of Hilbert space automorphisms of V which commute with the actions of both A and A'. This is the intersection $A \cap A' = Z(A)$. It follows that the center of A is a Morita invariant: any Morita equivalence between von Neumann algebras A and B induces an isomorphism $Z(A) \simeq Z(B)$.

Corollary 9. *The condition of being a factor is invariant under Morita equivalence. That is, if A and B are Morita equivalent, then A is a factor if and only if B is a factor.*

Proposition 10. *Let $A \subseteq B(V)$ be a von Neumann algebra. A von Neumann algebra B is Morita equivalent to A if and only if it can be realized as a subalgebra of some $B(W)$ such that B' is isomorphic to A'.*

Proof. Suppose $\mathrm{Rep}(A)$ is equivalent to $\mathrm{Rep}(B)$. The image of V under this equivalence is then a faithful representation of B whose endomorphism algebra (as a representation of B) is given by $\mathrm{Hom}_{\mathrm{Rep}(A)}(V, V) = A'$. This proves the "only if" direction. Conversely, suppose that we are given $B \subseteq B(W)$ and an isomorphism $A' \simeq B'$. Then we can regard W as a B-A'^{op} bimodule, which determines a functor $\mathrm{Rep}(A) \to \mathrm{Rep}(B)$. Similarly, we can regard V as an A-B'^{op} bimodule which determines a functor $\mathrm{Rep}(B) \to \mathrm{Rep}(A)$. It is easy to see that these functors are mutually inverse. □

Corollary 11. *Let A be a von Neumann algebra. The following conditions are equivalent:*

(1) *A is Morita equivalent to $\mathbf{C}$.*

(2) *There exists a Hilbert space V such that $A \simeq B(V)$.*

Proof. Regard $\mathbf{C}$ as a subalgebra of $B(\mathbf{C})$, so that its commutant is again $\mathbf{C}$. Applying Proposition 10, we see that A is Morita equivalent to $\mathbf{C}$ if and only if there exists an embedding $A \hookrightarrow B(V)$ whose commutant is $\mathbf{C}$. Since $A = A''$, this is equivalent to the condition that $A = B(V)$. □

Definition 12. We say that a von Neumann algebra A is a *type I factor* if it satisfies the equivalent conditions of Corollary 11.