Math 261y: von Neumann Algebras (Lecture 20)

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Let X be a standard finite measure space, fixed throughout this lecture. In the last lecture, we saw that the category of ultraweakly continuous, separable representations of $L^\infty(X)$ can be identified with the category of measurable fields of Hilbert spaces on X. In this lecture, we will exploit this equivalence to study von Neumann algebras containing $L^\infty(X)$ in their center.

Definition 1. Let $(\{V_x\}, V_{\text{meas}})$ be a measurable field of Hilbert spaces on X. A *field of von Neumann algebras* on $(\{V_x\}, V_{\text{meas}})$ is a specification, for each $x \in X$, of a von Neumann algebra $A_x \subseteq B(V_x)$. In this case, we let $\int_X A_x$ denote the collection of all (equivalence classes of) bounded maps of measurable fields $\{F_x : V_x \to V_x\}$ such that $F_x \in A_x$ for almost every x. We will identify $\int_X A_x$ with a set of bounded operators on $V_{\text{meas}}^{(2)}$.

We say that a field of von Neumann algebras $\{A_x\}$ is *measurable* if there exists a countable collection $F^1, F^2, \ldots \in \int_X A_x$ such that, for almost every $x \in X$, the operators F_x^i generate A_x as a von Neumann algebra. In this case, we will say that the F^i form a *generating sequence* for $\{A_x\}$.

The key technical result we will need is the following:

Theorem 2. *Let $(\{V_x\}, V_{\text{meas}})$ be a measurable field of Hilbert spaces on X, let $\{A_x \subseteq B(V_x)\}$ be a field of von Neumann algebras, and let $\{A'_x\}$ be their commutants. If $\{A_x\}$ is a measurable field of von Neumann algebras, then $\{A'_x\}$ is also measurable.*

Let us assume for simplicity that the field $\{V_x\}$ is constant: that is, that there is a fixed Hilbert space V such that $V_x = V$ for each $x \in X$, and V_{meas} is the set of measurable maps from X into V (this can always be achieved after partitioning X into countably many measurable subsets). Since X is standard, we may assume without loss of generality that X is an interval, equipped with some Borel measure (given by Lebesgue measure together with at most countably many atoms).

Let $B(V)_{\leq 1}$ denote the unit ball of $B(V)$, and regard $B(V)_{\leq 1}$ as endowed with the strong topology. Then $B(V)_{\leq 1}$ can be identified with a subspace of a product of countably many copies of $V_{\leq 1}$; in particular, $B(V)_{\leq 1}$ is a complete separable metric space. Moreover, the multiplication map

$$B(V)_{\leq 1} \times B(V)_{\leq 1} \to B(V)_{\leq 1}$$

is measurable. Choose a generating sequence $\{F^i\}$ for the field $\{A_x\}$. Rescaling, we may assume that $F_x^i \in B(V)_{\leq 1}$ for each i and each $x \in X$. Removing a set of measure zero from X, we can assume that each of the maps $F^i : X \to B(V)_{\leq 1}$ is Borel measurable. It follows that for each i, the map

$$X \times B(V)_{\leq 1} \to B(V)_{\leq 1}$$

$$(x, G) \mapsto F_x^i G - G F_x^i$$

is Borel measurable. Consequently, the inverse image of 0 under this map is Borel. Passing to the intersection over i, we deduce that the set

$$K = \{(x, G) \in X \times B(V)_{\leq 1} : G \in A'_x\}$$

is Borel.

Since $B(V)_{\leq 1}$ is a separable metric space, it has a countable basis $\{U_j\}_{j\geq 0}$. Each of the sets $K \cap U_j$ is Borel. We now need the following fact from descriptive set theory:

Theorem 3. *Let X and Y be complete, separable metric spaces, and suppose that X is equipped with a Borel measure μ. Let $p : X \times Y \to X$ denote the projection, and let $Z \subseteq X \times Y$ be a Borel subset, and assume Y is nonempty. Then*

(a) The image $p(Z)$ is μ-measurable.

(b) There exists a μ-measurable function $s : p(Z) \to Y$ such that, for each $x \in p(Z)$, $(x, s(x)) \in Z$.

Remark 4. In the situation of Theorem 3, the set $p(Z)$ need not be Borel. Recall that if X is a complete separable metric space, then a subset $X_0 \subseteq X$ is said to be *analytic* (or *Suslin*) if it is the continuous image of another complete separable metric space. Analytic sets need not be Borel, but one can show that they are μ-measurable for any Borel measure μ on X: that is, we can always find Borel sets B_- and B_+ such that

$$B_- \subseteq X_0 \subseteq B_+$$

and the difference $B_+ - B_-$ has μ-measure zero.

Let us apply Theorem 3 to our situation. We take $Y = B(V)_{\leq 1}$, and Z to be the subset $(X \times U_j) \cap K = U_j \cap \{(x, G) : G \in A'_x\}$. Let X_j denote the image of Z in X. Theorem 3 implies that each X_j is μ-measurable, and that there exist measurable maps $G^j : X_j \to B(V)_{\leq 1}$ such that $G^j_x \in U_j \cap A'_x$ for each $x \in X_j$. Let us extend the definition of G^j by setting $G^j_x = 0$ if $x \notin X_j$. For each j, we see that $\{G^j_x\}$ is a bounded map of measurable fields from $\{V_x\}$ to itself, belonging to $\int_X A'_x$. We claim that the G^j form a generating sequence for $\{A'_x\}$ (so that $\{A'_x\}$ is a measurable field of von Neumann algebras). To prove this, choose $x \in X$ and $G \in A'_x$; we wish to show that G belongs to the von Neumann algebra generated by the operators G^j_x. Scaling G, we may assume that $G \in B(V)_{\leq 1}$. Since this von Neumann algebra is strongly closed, it will suffice to show that for each strong neighborhood U of G in $B(V)_{\leq 1}$ contains G^j_x for some j. Without loss of generality, we may assume that U belongs to our countable basis; write $U = U_j$. Then $G \in U_j \cap A'_x$, so that $x \in X_j$. It follows that $G^j_x \in U \cap A'_x$, as desired. This completes the proof of Theorem 2.

Corollary 5. *Let $(\{V_x\}, V_{\text{meas}})$ be a measurable field of Hilbert spaces on X and let $\{A_x \subseteq B(V_x)\}$ be a measurable field of von Neumann algebras. Then $\int A_x$ is a von Neumann subalgebra of $V^{(2)}_{\text{meas}}$.*

Proof. Choose a generating sequence G^j for the field $\{A'_x\}$. Without loss of generality, we may assume that this sequence is closed under taking adjoints, so that the commutant of the sequence $\{G^j\}$ is a von Neumann algebra on $V^{(2)}_{\text{meas}}$.

We will complete the proof by showing that an operator $F : V^{(2)}_{\text{meas}} \to V^{(2)}_{\text{meas}}$ belongs to $\int A_x$ if and only if F commutes with each G^j and with the action of $L^\infty(X)$. The second of these assumptions implies that F arises from a bounded maps of fields $\{F_x : V_x \to V_x\}$, and the first assumption implies that for F_x commutes with G^j_x almost everywhere. Since the G^j_x generate A'_x almost everywhere, this implies that $F_x \in A''_x = A_x$ almost everywhere: that is, $F \in \int A_x$. □

Proposition 6. *Let $(\{V_x\}, V_{\text{meas}})$ be a measurable field of Hilbert spaces on X, and let $\{A_x\}$ and $\{B_x\}$ be measurable fields of von Neumann algebras on the field $\{V_x\}$. The following conditions are equivalent:*

(1) The von Neumann algebras $\int_X A_x$ and $\int_X B_x$ coincide.

(2) We have $A_x = B_x$ for almost every $x \in X$.

Proof. Let $\{F^i\}_{i\geq 0}$ be a generating sequence for $\{A_x\}$, so that $F^i \in \int_X A_x$. Thus $F^i \in \int_X B_x$. It follows that $F^i_x \in B_x$ for almost every x. Since the F^i_x generate A_x almost everywhere, we get $A_x \subseteq B_x$ for almost every x. The same arguments shows that $B_x \subseteq A_x$ for almost every x. □

We say that two measurable fields of von Neumann algebras $\{A_x\}$ and $\{B_x\}$ are *equivalent* if $A_x = B_x$ for almost every x.

Proposition 7. *Let $(\{V_x\}, V_{\mathrm{meas}})$ be a measurable field of Hilbert spaces on X. The construction*

$$\{A_x\} \mapsto \int_X A_x$$

induces a bijection from the set of equivalence classes of measurable fields of von Neumann algebras on $\{V_x\}$ with the set of von Neumann subalgebras of $B(V^{(2)}_{\mathrm{meas}})$, which contain the image of $L^\infty(X)$ in their center.

Proof. Let $\{A_x\}$ be a measurable field of von Neumann algebras on X. Theorem 2 implies that $\int_X A_x$ is a von Neumann algebra. It is clear that $\int_X A_x$ contains $\int_X \mathbf{C} \simeq L^\infty(X)$ in its center, and the injectivity of the construction $\{A_x\} \mapsto \int_X A_x$ follows from Proposition 6. It remains to prove surjectivity. Let $A \subseteq B(V^{(2)}_{\mathrm{meas}})$ be a von Neumann algebra containing (the image of) $L^\infty(X)$ in its center. Since $V^{(2)}_{\mathrm{meas}}$ is separable, the von Neumann algebra A' is separable. In particular, we can choose a countable sequence of operators $F^1, F^2, \ldots$ which is ultraweakly dense in A. Each F^i commutes with the action of $L^\infty(X)$, and so comes from a bounded map of fields $\{F^i_x\}$. Let A_x denote the von Neumann subalgebra of $B(V_x)$ generated by the F^i_x. By construction $\{A_x\}$ is a measurable field of von Neumann algebras. Since $F^i_x \in A_x$ for all $x \in X$, we get $F^i \in \int_X A_x$. Since the F^i are ultraweakly dense in A and $\int_X A_x$ is a von Neumann algebra, we conclude that $A \subseteq \int_X A_x$.

We now complete the proof by showing that $\int_X A_x \subseteq A$. Since A is a von Neumann algebra, we have $A = A''$. It will therefore suffice to show that operators in $\int_X A_x$ commute with operators belonging to the commutant A'. Let $G \in A'$. Since the image of $L^\infty(X)$ is contained in A, we see that G arises from a bounded map of fields $\{G_x : V_x \to V_x\}$. Since G commutes with each F^i, we deduce that the operators G_x and F^i_x commute for almost every $x \in X$. Thus $G_x \in A'_x$ for almost every x, from which it follows immediately that G commutes with every operator belonging to $\int_X A_x$. □

Proposition 8. *Let $(\{V_x\}, V_{\mathrm{meas}})$ be a measurable field of Hilbert spaces on X, and let $\{A_x\}$ be a measurable field of von Neumann algebras on the field $\{V_x\}$. Then we have*

$$(\int_X A_x)' = \int_X A'_x.$$

Proof. If $F \in \int_X A_x$ and $G \in \int_X A'_x$, then F and G come from bounded maps of fields $\{F_x, G_x : V_x \to V_x\}$ which commute for almost every $x \in X$¡ from which we deduce that F and G commute. This proves that $\int_X A'_x \subseteq (\int_X A_x)'$. Let us prove the reverse inclusion. Since $\int_X A_x$ commutes with $L^\infty(X)$, we have $L^\infty(X) \subseteq (\int_X A_x)'$. Since $\int_X A_x$ contains $L^\infty(X)$, we see that $L^\infty(X)$ belongs to the center of $(\int_X A_x)'$. We may therefore invoke Proposition 7 to write $(\int_X A_x)' = \int_X B_x$ for some measurable field of von Neumann algebras B_x. To prove the inclusion $\int_X B_x \subseteq \int_X A'_x$, it will suffice to show that B_x is contained in A'_x for almost every x.

Since $\{A_x\}$ and $\{B_x\}$ are measurable fields, we can choose generating sequences

$$F^i \in \int_X A_x \qquad G^j \in \int_X B_x,$$

coming from bounded maps of fields $\{F^i_x, G^j_x : V_x \to V_x\}$. Since $\int_X A_x$ and $\int_X B_x$ are commutants, the operators F^i and G^j commute. It follows that for every pair of integers i, j, the operators F^i_x and G^j_x commute for almost every $x \in X$. Since the F^i_x generate A_x and the G^j_x generate B_x, we conclude that $B_x \subseteq A'_x$ for almost every $x \in X$, as desired. □

Corollary 9. *Let $(\{V_x\}, V_{\mathrm{meas}})$ be a measurable field of Hilbert spaces on X, and let $\{A_x\}$ be a measurable field of von Neumann algebras on the field $\{V_x\}$. Let $Z(A_x)$ denote the center of A_x. Then $\{Z(A_x)\}$ is a measurable field of von Neumann algebras, and we have*

$$Z(\int_X A_x) = \int_X Z(A_x).$$

Proof. For each $x \in X$, let $A_x \vee A'_x$ denote the smallest von Neumann subalgebra of $B(V_x)$ containing A_x and A'_x. Then the field $x \mapsto A_x \vee A'_x$ is measurable (just take a union of generating sequences for A_x and A'_x). Note that

$$Z(A_x) = A_x \cap A'_x = A''_x \cap A'_x = (A'_x \vee A_x)'.$$

Applying Theorem 2, we see that $\{Z(A_x)\}$ is a measurable field of von Neumann algebras. Moreover, it is the largest measurable field of von Neumann algebras contained in both $\{A_x\}$ and $\{A'_x\}$. Since the one-to-one correspondence of Proposition 7 preserves orderings, we see that

$$\int_X Z(A_x)$$

is the intersection of the von Neumann algebras $\int_X A_x$ and $\int_X A'_x$. Using Proposition 8, we can write $\int_X A'_x = (\int_X A_x)'$, so that

$$\int_X Z(A_x) = \int_X A_x \cap (\int_X A_x)' = Z(\int_X A_x).$$

$\square$

Corollary 10. *Let $(\{V_x\}, V_{\mathrm{meas}})$ be a measurable field of Hilbert spaces on X, and let $\{A_x\}$ be a measurable field of von Neumann algebras on the field $\{V_x\}$. The following conditions are equivalent:*

(1) *The canonical map $L^\infty(X) \to Z(\int_x A_x)$ is an isomorphism.*

(2) *For almost every $x \in X$, we have $Z(A_x) = \mathbf{C}$.*

Recall that a von Neumann algebra A is said to be a *factor* if $Z(A) = \mathbf{C}$. If A is an arbitrary separable von Neumann algebra, we can realize $Z(A)$ as $L^\infty(X)$ for some standard measure space X, and embed A into $B(V)$ some some separable Hilbert space V. It follows from the results of the last two lectures that we can identify V with the space of square-integrable sections of a measurable field of Hilbert spaces $(\{V_x\}, V_{\mathrm{meas}})$ on X. It follows from Proposition 7 and Corollary 10 that there exists a measurable field of von Neumann algebras $\{A_x \subseteq B(V_x)\}$ such that $A = \int_X A_x$, where A_x is a factor for almost every $x \in X$. Consequently, the study of general von Neumann algebras can in some sense be reduced to the study of factors, which we will take up in the next lecture.