Math 261y: von Neumann Algebras (Lecture 2)

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Let us begin by setting up some terminological conventions which will we will use in this course. We will always work over the field $\mathbf{C}$ of complex numbers. By an *algebra* we will mean an associative ring A with a map $\mathbf{C} \rightarrow Z(A)$ (where $Z(A)$ is the center of A).

Let A be an algebra. A *norm* on A is a function

$$|| \, || : A \rightarrow \mathbb{R}_{\geq 0}$$

satisfying the following axioms:

$$||x|| = 0 \text{ if and only if } x = 0$$

$$||x + y|| \leq ||x|| + ||y||$$

$$||\lambda x|| = |\lambda| ||x|| \text{ if } \lambda \in \mathbf{C}$$

$$||xy|| \leq ||x|| \, ||y||$$

Every norm on A determines a metric d, given by $d(x, y) = ||x - y||$. We will say that A is *complete* if is complete with respect to this metric. A *Banach algebra* is a complete normed algebra.

Remark 1. Unless otherwise specified, we will always assume that our algebras are equipped with a unit. We will encounter nonunital algebras as well. In this case, some of the above definitions need to be modified.

Example 2. Let X be a compact topological space. Then the algebra $C^0(X)$ of continuous functions on X is a Banach algebra.

Example 3. Let V be a Hilbert space (or, more generally, a Banach space) and let $B(V)$ be the algebra of bounded operators on V. Then $B(V)$ is a Banach algebra with respect to the *operator norm*, given by

$$||f|| = \sup\{||f(v)||, 0\}_{v \in V, ||v||=1}$$

Example 4. Let G be a locally compact group, and let $L^1(G)$ denote the Banach space of integrable functions on G (with respect to Haar measure). The operation of convolution equips $L^1(G)$ with the structure of a Banach algebra. This Banach algebra is not unital unless G is discrete. However, it can be embedded in the larger *unital* Banach algebra $M(G)$ of finite Borel measures on G (where multiplication is again given by convolution, and the identity element is given by the point measure supported at the identity of G.

Proposition 5. *Let A be a Banach algebra. Then the collection of invertible elements of A is open.*

Proof. Let $x \in A$ be invertible; we wish to show that x has an open neighborhood in A consisting of invertible elements. Multiplying by x^{-1} (which is a homeomorphism of A with itself), we can reduce to the case $x = 1$. In this case, we wish to show that $1 + y$ is invertible provided that the norm of y is sufficiently small. The inverse of $1 + y$ is given by the power series

$$1 - y + y^2 - y^3 + y^4 - \cdots ,$$

which converges for $||y|| < 1$. □

Definition 6. Let A be a Banach algebra and let $x \in A$. The *spectrum of* x is the set of complex numbers λ such that $x - \lambda$ is not invertible. We will denote the spectrum of x by $\sigma(x)$. It follows from Proposition 5 that $\sigma(x)$ is a closed subset of the complex numbers.

Proposition 7. *Let A be a nonzero Banach algebra and let $x \in A$. Then the spectrum $\sigma(x)$ is nonempty.*

Proof. Suppose otherwise; then the difference $\lambda - x$ is invertible for each $\lambda \in \mathbf{C}$. Let $\phi : A \to \mathbf{C}$ be a continuous linear functional. One can show that the map

$$\lambda \mapsto \phi(\frac{1}{\lambda - x})$$

is a holomorphic function on the complex numbers, which is bounded (in fact, it vanishes at infinity). It follows from complex analysis that this function is constant. Since ϕ is arbitrary, we deduce that the function $\lambda \mapsto \frac{1}{\lambda - x}$ is constant, so that the function $\lambda \mapsto \lambda - x$ is constant. This is only possible if $A = 0$. $\square$

Corollary 8 (Gelfand-Mazur)**.** *Let A be a Banach division algebra. Then $A \simeq \mathbf{C}$.*

Proof. Let $x \in A$. Since $A \neq 0$, the spectrum $\sigma(x)$ is nonzero. Choose $\lambda \in \sigma(x)$, so that $\lambda - x$ is not invertible. Since A is a division algebra, we deduce that $\lambda - x = 0$, so that $x = \lambda$. It follows that every element of A is a scalar. $\square$

We can regard Corollary 8 as a Banach-algebraic analogue of Hilbert's Nullstellensatz:

Corollary 9. *Let A be a commutative Banach algebra, and let $\mathfrak{m}$ be a maximal ideal in A. Then the quotient $A/\mathfrak{m}$ is isomorphic to $\mathbf{C}$.*

Proof. Let $\overline{\mathfrak{m}}$ be the closure of $\mathfrak{m}$. Proposition 5 implies that there is an open neighborhood U of the identity element $1 \in A$ consisting of invertible elements, which cannot intersect $\mathfrak{m}$. It follows that $1 \notin \overline{\mathfrak{m}}$, so that $\overline{\mathfrak{m}}$ is a proper ideal of A. The maximality of $\mathfrak{m}$ implies that $\mathfrak{m} = \overline{\mathfrak{m}}$, so that $\mathfrak{m}$ is closed. Then the quotient $A/\mathfrak{m}$ inherits a complete norm, and is therefore a Banach algebra in its own right. Since $\mathfrak{m}$ is maximal, $A/\mathfrak{m}$ is a field. Applying Corollary 8, we deduce that $A/\mathfrak{m} \simeq \mathbf{C}$. $\square$

Note that if A is a Banach algebra, the spectrum of an element $x \in A$ cannot contain any complex number $\lambda > ||x||$, since for $\lambda > ||x||$ the formal series

$$1 + \frac{x}{\lambda} + \frac{x^2}{\lambda^2} + \frac{x^3}{\lambda^3} + \cdots$$

will converge to an inverse of $1 - \frac{x}{\lambda}$. It follows that $\sigma(x)$ is a closed and bounded subset of $\mathbf{C}$, hence compact.

Definition 10. Let A be a Banach algebra and let $x \in A$. The *spectral radius* of x is given by

$$\rho(x) = \max\{|\lambda| : \lambda \in \sigma(x)\}.$$

Our first nontrivial result is the following:

Theorem 11 (Gelfand's Spectral Radius Formula)**.** *Let A be a Banach algebra and let $x \in A$. Then*

$$\rho(x) = \limsup ||x^n||^{\frac{1}{n}}$$

Remark 12. One can say more: for example, the sequence $||x^n||^{\frac{1}{n}}$ converges to $\rho(x)$. We will not need this.

Proof. It will suffice to show that for every positive real number ϵ, we have

$$\epsilon < \rho(x) \Leftrightarrow \epsilon < \limsup ||x^n||^{\frac{1}{n}}.$$

- Suppose first that $\limsup ||x^n||^{\frac{1}{n}} \leq \epsilon$. We wish to show that $\rho(x) \leq \epsilon$: that is, that $x - \lambda$ is invertible whenever $|\lambda| > \epsilon$. Rescaling if necessary, we may assume that $\lambda = 1$ (so that $\epsilon < 1$). Choose $\epsilon < \delta < 1$. Since $\limsup ||x^n||^{\frac{1}{n}} \leq \epsilon$, we conclude that $||x^n||^{\frac{1}{n}} \leq \delta$ for almost every integer n. It follows that the power series $1 + x + x^2 + \cdots$ converges absolutely in A to an inverse of $1 - x$.

- Suppose that $\limsup ||x^n||^{\frac{1}{n}} > \epsilon$; we wish to prove that $\rho(x) > \epsilon$. Rescaling if necessary, we may assume that $\epsilon = 1$. Choose δ with $1 < \delta < \delta^2 < \limsup ||x^n||^{\frac{1}{n}}$, so that $||x^n|| \geq \delta^{2n}$ for infinitely many values of n. It follows that the sequence $1, \frac{x}{\delta}, \frac{x^2}{\delta^2}, \ldots$ is unbounded in A. Let $A^\vee$ denote the dual space of of A, and think of the elements x^i as linear functionals on $A^\vee$. Using the uniform boundedness principle, we deduce that there exists a continuous linear functional $\phi \in A^\vee$ such that the sequence $\phi(1), \phi(\frac{x}{\delta}), \phi(\frac{x^2}{\delta^2}), \ldots$ is unbounded. Consider the function $\lambda \mapsto \phi(\frac{\lambda}{\lambda - x})$, which is well-defined an holomorphic for $\lambda \notin \sigma(x)$. For $|\lambda|$ large, this function is given by

$$\phi(1) + \phi(\frac{x}{\lambda}) + \phi(\frac{x^2}{\lambda^2}) + \cdots.$$

  If $\rho(x) \leq 1$, then complex analysis implies that this series converges absolutely $|\lambda| > 1$. Since it does not converge absolutely for $\lambda = \delta$, we obtain a contradiction.

□

Corollary 13. *Let A be a Banach algebra and let $\chi : A \to \mathbf{C}$ be an algebra homomorphism. Then χ has norm ≤ 1. In particular, χ is continuous.*

Proof. Let $x \in A$. We wish to show that $|\chi(x)| \leq ||x||$. Assume otherwise; then $|\chi(x)| > ||x|| \geq \rho(x)$, so that $\chi(x) - x$ is invertible in A. This is a contradiction, since the image of $\chi(x) - x$ in $\mathbf{C}$ is zero. □

Definition 14. Let A be a commutative Banach algebra. The *spectrum* of A is the collection of algebra homomorphisms $\chi : A \to \mathbf{C}$ (automatically continuous, by Corollary 13). We denote the spectrum of A by $\operatorname{Spec} A$.

Remark 15. Let A be a commutative Banach algebra and let $x \in A$. A complex number λ belongs to the spectrum $\sigma(x)$ if and only if $\lambda - x$ is not invertible: that is, if and only if $\lambda - x$ generates a nontrivial ideal in A. Since every nontrivial ideal in A is contained in a maximal ideal, we see that $\lambda \in \sigma(x)$ if and only if $\lambda - x$ is contained in a maximal ideal of A, which is then the kernel of an algebra homomorphism $\chi : A \to \mathbf{C}$. We deduce that $\sigma(x) = \{\chi(x) : \chi \in \operatorname{Spec} A\}$.

Let us regard $\operatorname{Spec} A$ as a subset of the product

$$\prod_{x \in A} \sigma(x).$$

It is easy to see that $\operatorname{Spec} A$ is closed (with respect to the product topology on $\prod_{x \in A} \sigma(x)$, and therefore inherits the structure of a compact Hausdorff space.

We now study algebras with a bit more structure.

Definition 16. A $*$-*algebra* is an algebra A equipped with a map $x \mapsto x^*$ satisfying the following axioms:

$$(xy)^* = y^* x^* \qquad (x + y)^* = x^* + y^* \qquad \lambda^* = \overline{\lambda} \text{ if } \lambda \in \mathbf{C} \qquad x^{**} = x$$

A C^*-*algebra* is a Banach $*$-algebra whose norm satisfies the following identity:

$$||x||^2 = ||x^* x||$$

Example 17. Let X be a compact Hausdorff space. Then $C^0(X)$ is a C^*-algebra, with involution given by complex conjugation of functions.

Example 18. Let V be a Hilbert space. Then the algebra of bounded operators $B(V)$ is a $*$-algebra, where we take f^* to be the *adjoint* of f, characterized by the formula

$$(f^*v, w) = (v, fw).$$

In fact, it is a C^*-algebra. The inequality $||x^*x|| \leq ||x^*||\,||x|| = ||x||\,||x|| = ||x||^2$ is obvious. To prove the reverse inequality $||x||^2 \leq ||xx^*||$, it will suffice to show that for any positive real number $\epsilon < ||x||$, there exists a vector $v \in V$ with $||v|| = 1$ and $||x^*xv|| > \epsilon^2$. Using the definition of $||x||$, we can choose v such that $||xv|| > \epsilon$, so that

$$\epsilon^2 > ||xv||^2 = (xv, xv) = (x^*xv, v) \leq ||x^*xv||.$$

Example 19. Let G be a unimodular locally compact group and regard $L^1(G)$ as a nonunital Banach algebra via convolution. Then $L^1(G)$ is equipped with the structure of a $*$-algebra, given by

$$f^*(g) = \overline{f(g^{-1})}.$$

It is generally not a C^*-algebra.

Notation 20. Let A be a $*$-algebra. We say that an element $x \in A$ is *Hermitian* or *self-adjoint* if $x = x^*$. We say that x is *skew-Hermitian* or *skew-adjoint* if $x^* = -x$. Every element $x \in A$ admits a unique decomposition $x = \Re(x) + \Im(x)$, where $\Re(x) = \frac{x+x^*}{2}$ is self-adjoint and $\Im(x) = \frac{x-x^*}{2}$ is skew-adjoint.

We say that an element $x \in A$ is *normal* if x and x^* commute: equivalently, x is normal if $\Re(x)$ and $\Im(x)$ commute.

Proposition 21. *Let A be a C^*-algebra and let $x \in A$ be a normal element. Then $||x^n|| = ||x||^n$ for every positive integer n.*

Proof. It will suffice to show that $||x^n||^2 = ||x||^{2n}$. Applying the C^*-identity, we can rewrite this as $||(x^n)^*x^n|| = ||x^*x||^n$. Since x is normal, the left hand side can be rewritten $||(x^*x)^n||$. We may therefore replace x by x^*x and thereby reduce to the case where x is Hermitian. In this case, the C^*-identity gives $||x^2|| = ||x||^2$. Iterating this argument, we obtain

$$||x^{2^k}|| = ||x||^{2^k}.$$

Choose an integer m such that $m + n$ is a power of 2. We then have

$$||x^{m+n}|| = ||x^m x^n|| \leq ||x^m||\,||x^n|| \leq ||x||^m\,||x||^n = ||x||^{m+n}.$$

Since equality holds, we must have equality throughout. Assuming $x \neq 0$, this gives

$$||x^m|| = ||x||^m \qquad ||x^n|| = ||x||^n.$$

□

Corollary 22. *Let A be a C^* algebra and let $x \in A$ be a normal element. Then the spectral radius $\rho(x)$ coincides with the norm $||x||$.*

Proof. Combine the spectral radius formula (Theorem 11) with Proposition 21. □

For any commutative Banach algebra A, each element $x \in A$ determines a continuous map $\operatorname{Spec} A \to \mathbf{C}$, given by $\chi \mapsto \chi(x)$. This map is an algebra homomorphism, called the *Gelfand transform*.

Proposition 23. *Let A be a commutative C^*-algebra. Then the Gelfand transform $u : A \to C^0(\operatorname{Spec} A)$ is an isomorphism of C^*-algebras.*

Proof. We first show that u is a map of $*$-algebras. Equivalently, we claim that every character $\chi : A \to \mathbf{C}$ satisfies $\chi(x^*) = \overline{\chi(x)}$. It will suffice to show that χ carries Hermitian elements $x \in A$ to real numbers. Define $f : \mathbb{R} \to A$ by the formula

$$f(t) = e^{itx} = \sum_n \frac{(itx)^n}{n!}.$$

Then f satisfies $f(t)^{-1} = f(-t) = f(t)^*$, so that the C^*-identity gives

$$||f(t)||^2 = ||f(t)^* f(t)|| = 1.$$

Since χ is continuous and has norm ≤ 1 (Corollary 13), we obtain

$$1 \geq |\chi f(t)| = e^{it\chi(x)}.$$

Since this is true for both positive and negative values of t, we must have $\chi(x) \in \mathbb{R}$.

We now note that the Gelfand transform u is isometric: for $x \in A$ we have

$$||u(x)|| = \sup\{\chi \in \operatorname{Spec} A : |\chi(x)|\} = \rho(x) = ||x||$$

by Corollary 22. It follows that u is an isomorphism from A onto a closed $*$-subalgebra of $C^0(\operatorname{Spec} A)$. This subalgebra separates points: if $\chi, \chi' \in \operatorname{Spec} A$ are distinct, then we can choose $x \in A$ such that $\chi(x) \neq \chi'(x)$. Applying the Stone-Weierstrass theorem, we deduce that the image of u is the whole of $C^0(\operatorname{Spec} A)$, so that u is an isomorphism. □

Corollary 24. *Every commutative C^*-algebra is isomorphic to $C^0(X)$ for some compact Hausdorff space X. Moreover, we can canonically recover X as the spectrum* $\operatorname{Spec} A$.