# Math 261y: von Neumann Algebras (Lecture 19)

October 17, 2011

Let $X$ be a standard measure space, fixed throughout this lecture.

Our main goal in this lecture is to complete the proof we began in the previous lecture, establishing an equivalence between measurable fields of Hilbert spaces on $X$ and separable representations of $L^\infty(X)$.

Suppose we are given measurable fields $(\{V_x\}, V_{\text{meas}})$ and $(V'_x, V'_{\text{meas}})$, and a bounded operator $F : V^{(2)}_{\text{meas}} \to V'^{(2)}_{\text{meas}}$. which commutes with the action of $L^\infty(X)$. Let $C = ||F||$, let $f_i$ be a normalized generating sequence in $V_{\text{meas}}$. We claim that for almost every $x \in X$, the construction

$$v \mapsto \sum_{i \geq 1} (v, f_i(x))_x F(f_i)(x)$$

determines a bounded operator $F_x : V_x \to V'_x$, having operator norm $\leq C$. This map is evidently well-defined on the linear span of the vectors $f_i(x)$ in $V_x$ To prove this operator has norm $\leq C$, it will suffice to verify this on the finite-dimensional subspaces spanned by $\{f_i(x)\}_{1 \leq i \leq n}$ for each $n \geq 0$. It will suffice to show that for almost every $x \in X$, the map

$$\mathbf{C}^n \to V'_x$$

$$(z_1, \ldots, z_n) \mapsto \sum_{1 \leq i \leq n} z_i F(f_i)(x)$$

has norm $\leq C$. Note that if this condition is violated for some $x \in X$, then there exists a vector $\vec{z} = (z_1, \ldots, z_n)$ lying in the unit ball of $\mathbf{C}^n$ such that $|| \sum_{1 \leq i \leq n} z_i F(f_i)|| > C$. The collection of such vectors $\vec{z}$ forms an open subset of the unit ball of $\mathbf{C}^n$. Choose a countable dense subset $S$ of the unit ball, and for each $\vec{z} \in S$ set $X_{\vec{z}} = \{x \in X : | \sum z_i F(f_i)(x)| > C\}$. We wish to show that $\bigcup_{\vec{z} \in S} X_{\vec{z}}$ has measure zero. Assume otherwise; then $Y = X_{\vec{z}}$ has positive measure for some $\vec{z} = (z_1, \ldots, z_n) \in S$. Set $f = \chi_Y (\sum z_i f_i) \in V^{(2)}_{\text{meas}}$, where $\chi_Y$ denotes the characteristic function of $Y$. Then

$$F(f) = \chi_Y \sum z_i F(f_i)$$

has norm $> C$ at the points of $Y$. Thus

$$||F(f)||^2 > C^2 \mu(Y) > C^2 ||f||^2,$$

contradicting our assumption on the norm of $F$. This completes the proof that the construction

$$(\{V_x\}, V_{\text{meas}}) \mapsto V^{(2)}_{\text{meas}}$$

is fully faithful.

We now prove the essential surjectivity. Suppose that $V$ is a separable representation of $L^\infty(X)$. Choose an orthonormal basis $v_1, v_2, \ldots$ for $V$, and let $V_0$ be the subspace of $V$ consisting of finite linear combinations of the vectors $v_i$.

For $i, j \leq 1$, the construction

$$(\lambda \in L^\infty(X)) \mapsto (\lambda v_i, v_j)$$

is ultraweakly continuous, and therefore given by

$$\lambda \mapsto \int_X \lambda h_{i,j} d\mu$$

for some $h_{i,j} \in L^1(X)$. Note that $h_{i,j} = \overline{h_{j,i}}$. For each $x \in X$, we define an inner product $(,)_x$ on $V_0$ by the formula

$$(v_i, v_j)_x = h_{i,j}.$$

**Lemma 1.** *In the situation above, the inner product $(,)_x$ is positive semidefinite for almost every $x \in X$.*

*Proof.* It will suffice to show that, for each $n \geq 0$, the inner product $(,)_x$ is positive semidefinite on $W = \mathbf{C}v_1 + \cdots + \mathbf{C}v_n$ for almost every $x$. Let $Y$ be the set of those elements $x \in X$ such that $(,)_x$ is not positive semidefinite on $W$. Choose a countable dense subset $S \subseteq W$. For $x \in Y$, there exists a vector $w \in W$ such that $(w, w)_x < 0$. The collection vectors $w$ which satisfy this condition is open, so we may assume without loss of generality that $w \in S$. Thus

$$Y = \bigcup_{w \in S} Y_w,$$

where $Y_w = \{x \in X : (w, w)_x < 0\}$. It will therefore suffice to show that each $Y_w$ has measure zero. Assume otherwise and let $\chi$ denote the characteristic function of $Y_w$. Then

$$0 \leq (\chi w, w) = \int_X \chi(w, w)_x d\mu < 0,$$

a contradiction. □

Throwing out a set of measure zero, we may assume that each of the inner products $(,)_x$ is positive semidefinite; we let $V_x$ denote the Hilbert space obtained by completing $V_0$ with respect to the inner product $(,)_x$. In what follows, we will abuse notation by identifying elements of $V_0$ with their images in $V_x$. Let $\delta : V_0 \to \prod_{x \in X} V_x$ be the evident diagonal map. Let us say that a section $g \in \prod_{x \in X} V_x$ is *measurable* if the function $x \mapsto (g(x), v_i)_x$ is measurable for each $i \geq 1$. Let $V_{\text{meas}} \subseteq \prod_{x \in X} V_x$ be the collection of measurable sections. Since the functions $h_{i,j}$ are measurable, we conclude that $\delta(v_i) \in V_{\text{meas}}$ for $i \geq 1$. It follows that $(\{V_x\}, V_{\text{meas}})$ is a measurable field of Hilbert spaces on $X$, with a generating sequence given by the $\delta(v_i)$.

By construction, we have

$$(\delta(v_i), \delta(v_j)) = \int_X (v_i, v_j)_x d\mu = \int h_{i,j} d\mu = (v_i, v_j).$$

In particular, each $\delta(v_i)$ is square integrable, and the map $\delta : V_0 \to V^{(2)}_{\text{meas}}$ is isometric. The map $\delta$ extends uniquely to a map of (non-topologized) $L^\infty(X)$-modules

$$\delta^+ : L^\infty(X) \otimes_{\mathbf{C}} V_0 \to V^{(2)}_{\text{meas}}.$$

The $L^\infty(X)$-module structure on $V$ determines a map $L^\infty(X) \otimes_{\mathbf{C}} V_0 \to V$, and therefore a pre-inner product on $L^\infty(X) \otimes_{\mathbf{C}} V_0$. The vector space $L^\infty(X) \otimes_{\mathbf{C}} V_0$ is spanned by vectors of the form $\lambda v_i$, where $\lambda \in L^\infty(X)$. We compute

$$\begin{aligned}
(\delta^+(\lambda v_i), \delta^+(\lambda' v_j)) &= (\lambda \delta(v_i), \lambda' \delta(v_j)) \\
&= \int_X \lambda \overline{\lambda}' h_{i,j} d\mu \\
&= (\lambda \overline{\lambda}' v_i, v_j) \\
&= (\lambda v_i, \lambda v_j).
\end{aligned}$$

It follows that the map $\delta^+$ is an isometry. Since the map $L^\infty(X) \otimes_{\mathbf{C}} V_0 \to V$ has dense image, it exhibits $V$ as the Hilbert space completion of $L^\infty(X) \otimes_{\mathbf{C}} V_0$. It follows that $\delta^+$ factors through an isometry of Hilbert spaces $\overline{\delta} : V \to V^{(2)}_{\text{meas}}$, which is evidently a map of $L^\infty$-modules.

To complete the proof, it will suffice to show that $\overline{\delta}$ is an isomorphism. Since it is an isometry, it is injective. The image of $\overline{\delta}$ is an $L^\infty(X)$-submodule of $V^{(2)}_{\text{meas}}$ which contains the generating sequence $\delta(v_1), \delta(v_2), \ldots$, and is therefore dense in $V^{(2)}_{\text{meas}}$. Since this image is also closed (by virtue of the fact that $\overline{\delta}$ is an isometry), we conclude that $\overline{\delta}$ is surjective.