Math 261y: von Neumann Algebras (Lecture 17)

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Our first goal is to complete the proof we started in the previous lecture. We are reduced to the following:

Proposition 1. *Let (X, Σ, μ) be a finite measure space with no atoms, and suppose we are given a countable collection of measurable subsets $E_0, E_1, \ldots$ which generate the σ-algebra Σ/Σ_0. Then there exists a measure-preserving map $f : X \to [0, \mu(X)]$ such that each E_i is equivalent (modulo sets of measure zero) to $f^{-1}B_i$, for some Borel subset $B_i \subseteq [0, \mu(X)]$.*

Lemma 2. *Suppose we are given a finite collection $\mathcal{S} = \{S_0, S_1, \ldots, S_n\}$ of measurable subsets of X which are linearly ordered with respect to inclusion, and let $E \subseteq X$ be an arbitrary measurable subset of X. Then there exists a larger finite collection $\mathcal{S}'$ of measurable subsets of X, again linearly ordered with respect to inclusion, such that E belongs to the σ-algebra of subsets generated by $\mathcal{S}'$.*

Proof. Without loss of generality we may assume that

$$\emptyset = S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n = X.$$

Set $K_m = S_m - S_{m-1}$ for $1 \leq m \leq n$, so that the K_m are disjoint sets with union X. We now set $\mathcal{S}' = \{S_0, K_1 \cap E, S_1, S_1 \cup (K_2 \cap E), S_2, S_2 \cup (K_3 \cap E), \ldots, S_n\}$. □

Repeatedly applying Lemma 2, are we reduced to proving Proposition 1 in the case where the collection of sets $\{E_i\}$ is linearly ordered by inclusion (of course, this linear ordering need not coincide with the order in which we are enumerating the sets). Set $t_i = \mu(E_i)$. We now define the function $f : X \to [0, \mu(X)]$ by the formula $f(x) = \inf\{t_i : x \in E_i\}$. It is easy to see that f is a measurable function. We claim that f has the desired properties. First, we show that for each $i \geq 0$, the subsets E_i and $f^{-1}[0, t_i]$ differ by a set of measure zero. We clearly have $f(x) \leq t_i$ for $x \in E_i$; it will therefore suffice to show that the set $K = f^{-1}[0, t_i] - E_i$ has measure zero. Note that

$$K = \bigcap_{\epsilon > 0} \{x \in X : (\exists j)[(t_j < t_i + \epsilon) \text{ and } x \in E_j - E_i]\}.$$

Since the E_j's are linearly ordered, if $t_j < t_i + \epsilon$, then $\mu(E_j - E_i) < \epsilon$. It follows that $\{x \in X : (\exists j)[(t_j < t_i + \epsilon) \text{ and } x \in E_j - E_i]\}$ is the union of a linearly ordered collection of sets of measure $< \epsilon$, and therefore has measure $\leq \epsilon$. Thus $\mu(K) \leq \epsilon$ for every $\epsilon > 0$, which proves that $\mu(K) = 0$ as desired.

It remains to prove that f is measure preserving. We have just seen that $f^{-1}[0, t_i]$ is a set of measure $t_i = \mu(E_i)$ for every integer i. To prove that f preserves measure on arbitrary Borel sets, it will suffice to show that the t_i are dense in the interval $[0, \mu(X)]$. Suppose otherwise. Then there exists an interval $(a, b) \subseteq [0, \mu(X)]$ which does not contain any t_i. We may therefore partition the nonnegative integers into two subsets:

$$J_- = \{i : t_i \leq a\} \qquad J_+ = \{i : t_i \geq b\}.$$

Let $Y = \bigcap_{i \in J_+} E_i - \bigcup_{i \in J_-} E_i$. Then $\mu(Y) \geq b - a > 0$. By construction, every E_i either contains Y or is disjoint from Y. Since the sets E_i generate the σ-algebra Σ/Σ_0, we see that every measurable subset of X either contains Y (modulo sets of measure zero) or is disjoint from Y (modulo sets of measure zero). Then

Y is an atom, contradicting our assumption that X has no atoms. This completes the proof of Proposition 1.

We now turn to a new topic: the functional calculus of von Neumann and C^*-algebras. Suppose that A is a C^*-algebra containing a normal element x. Let $A_0 \subseteq A$ be the C^*-subalgebra of A generated by x. Since x is normal, A_0 is commutative, hence of the form $C^0(X)$ for some compact Hausdorff space X. We can regard x as defining a continuous map $u : X \to \mathbf{C}$, whose image is precisely the spectrum $\sigma(x)$. The pullback map

$$\rho : C^0(\sigma(x)) \to C^0(X) = A_0$$

is a map of C^*-algebras which carries the identity function on $\sigma(x)$ to X. Since A_0 is generated by x (as a C^*-algebra), we see that ρ is surjective, so that u is injective. Since X is compact and $\sigma(x)$ is Hausdorff, we conclude that u is a homeomorphism from X onto $\sigma(x)$. We have proven:

Proposition 3. *Let A be a C^*-algebra containing a normal element x. Then the C^*-subalgebra of A generated by x is (canonically) isomorphic to $C^0(\sigma(x))$.*

We will denote the composite map

$$C^0(\sigma(x)) \simeq A_0 \hookrightarrow A$$

by $f \mapsto f(x)$. This is known as the *continuous functional calculus*. It is easy to see that this construction has the following properties:

- (a) If f is the identity function $z \mapsto z$, then $f(x) = x$.
- (b) If $\overline{f}$ denotes the complex conjugate of the function f, then $\overline{f}(x) = f(x)^*$.
- (c) Given a pair of continuous functions $f, g : \sigma(x) \to \mathbf{C}$ and any scalar $\lambda \in \mathbf{C}$, we have

$$(f+g)(x) = f(x) + g(x) \qquad (fg)(x) = f(x)g(x) \qquad (\lambda f)(x) = \lambda f(x).$$

- (d) Given a sequence of functions $f_i : \sigma(x) \to \mathbf{C}$ which converge uniformly to f, we have $f(x) = \lim\{f_i(x)\}$ (where the limit is taken with respect to the norm topology on A).

Moreover, the continuous functional calculus is uniquely determined by this properties: according to the Stone-Weierstrass theorem, every continuous function $f : X \to \mathbf{C}$ is the limit of a uniformly convergent sequence of functions which are polynomials in z and $\overline{z}$.

If A is a von Neumann algebra, there is an even better version of the functional calculus. Let's begin by considering the case where A is a commutative von Neumann algebra, so that there exists an isomorphism $A \simeq L^\infty(X)$ for some measure space X.

Proposition 4. *Let $A = L^\infty(X)$, and let $g \in A$. Then the spectrum $\sigma(g)$ is the smallest closed subset $K \subseteq \mathbf{C}$ such that $g^{-1}(\mathbf{C} - K)$ has measure zero.*

Proof. For $\lambda \in \mathbf{C}$, we have $\lambda \notin \sigma(g)$ if and only if $f - \lambda$ is invertible in $L^\infty(X)$: that is, if and only if there exists an open neighborhood U containing λ such that $g^{-1}U$ has measure zero. It follows immediately that if $g^{-1}(\mathbf{C} - K)$ has measure zero, then $\sigma(x) \subseteq K$.

Without loss of generality, we can restrict U to range over some countable basis for the topology of $\mathbf{C}$. Then $\mathbf{C} - \sigma(g)$ is the union of a countable collection of open sets whose inverse images have measure zero, so that $g^{-1}(\mathbf{C} - \sigma(g))$ has measure zero. □

In particular, we see that if $g \in L^\infty(X)$, then (after modifying f on a set of measure zero) we can assume that g takes values in the closed set $\sigma(x) \subseteq \mathbf{C}$. If $f : \sigma(x) \to \mathbf{C}$ is a bounded Borel measurable function, then $f \circ g$ is a bounded measurable function on X, which determines another element of $A = L^\infty(X)$. It is easy to see that this is independent of the representative chosen for g.

We claim that the construction $g \mapsto g \circ f$ has the following properties:

(a) If f is the identity function $z \mapsto z$, then $f \circ g = g$.

(b) If $\overline{f}$ denotes the complex conjugate of the function f, then $\overline{f} \circ g = \overline{f \circ g} = (f \circ g)^*$.

(c) Given a pair of bounded Borel measurable functions $f, f' : \sigma(x) \to \mathbf{C}$ and any scalar $\lambda \in \mathbf{C}$, we have

$$(f + f') \circ g = (f \circ g) + (f' \circ g) \qquad (ff') \circ g = (f \circ g)(f' \circ g) \qquad (\lambda f) \circ g = \lambda(f \circ g).$$

(d) Given a uniformly bounded sequence of functions $f_i : \sigma(g) \to \mathbf{C}$ which converge pointwise to f, we have $f \circ g = \lim\{f_i \circ g\}$ (where the limit is taken with respect to the ultraweak topology on A).

Only the last statement requires proof. It is a consequence of the following assertion:

Lemma 5. *Let $A = L^\infty(X)$ be a commutative von Neumann algebra, and suppose that $\{g_i\}$ is a sequence of uniformly bounded measurable functions on X which converge pointwise to a function g. Then g is a limit of the sequence $\{g_i\}$ with respect to the ultraweak topology.*

Proof. The predual of the von Neumann algebra $L^\infty(X)$ is the Banach space $L^1(X)$. Since the ultraweak topology coincides with the weak $*$-topology, we are reduced to proving that for every $f \in L^1(X)$, the sequence of integrals

$$\int_X f g_i$$

converges to $\int_X fg$. This follows from the dominated convergence theorem. □

Note that properties (a) through (d) uniquely characterize the construction $f \mapsto f \circ g$, since every Borel measurable function $\sigma(g) \to \mathbf{C}$ can be obtained as a pointwise limit of uniformly bounded sequence of continuous functions on $\sigma(g)$ (and therefore, using the Stone-Weierstrass theorem, as a pointwise limit of a uniformly bounded sequence of functions which are polynomials in z and $\overline{z}$). This proves the following:

Theorem 6 (Borel Functional Calculus)**.** *Let A be a von Neumann algebra and let $x \in A$ be a normal element. Let S be the collection of all bounded Borel measurable functions on the spectrum $\sigma(x)$. Then there is a unique map $S \to A$, which we will denote by $f \mapsto f(x)$, with the following properties:*

(a) *If f is the identity function $z \mapsto z$, then $f(x) = x$.*

(b) *If $\overline{f}$ denotes the complex conjugate of the function f, then $\overline{f}(x) = f(x)^*$.*

(c) *Given a pair of bounded Borel-measurable functions $f, g : \sigma(x) \to \mathbf{C}$ and any scalar $\lambda \in \mathbf{C}$, we have*

$$(f + g)(x) = f(x) + g(x) \qquad (fg)(x) = f(x)g(x) \qquad (\lambda f)(x) = \lambda f(x).$$

(d) *Given a uniformly bounded sequence of functions $f_i : \sigma(x) \to \mathbf{C}$ which converge pointwise to f, we have $f(x) = \lim\{f_i(x)\}$ (where the limit is taken with respect to the ultraweak on A).*

Proof. By replacing A by the smallest von Neumann algebra containing x, we can reduce to the case where A is commutative, in which case $A \simeq L^\infty(X)$ and the result follows from the explicit construction given earlier. □

Remark 7. Using the uniqueness, one can show that the Borel functional calculus on von Neumann algebras extends the continuous functional calculus on C^*-algebras.

Remark 8. The functional calculus $f \mapsto f(x)$ is generally not defined if we assume only that f is Lebesgue measurable, rather than Borel measurable. Note also that $f(x)$ generally changes if we alter the definition of f on a set of measure zero.

Remark 9. In addition to the Borel functional calculus for von Neumann algebras and the continuous functional calculus for C^*-algebras, there is a *holomorphic* functional calculus which is defined in the setting of arbitrary Banach algebras.